1 For $-1 \leq x \leq 2$ and real y, define

$$H(x, y) = (x^2 - 2)^2 + (x - y)^2.$$

For a real number k, consider the statements

P : For every $x \in [-1, 2]$ there is a real y with $H(x, y) < k$.
Q : There is a real y such that $H(x, y) < k$ for every $x \in [-1, 2]$.

Which one of the following describes exactly the values of k for which P is true and Q is false?

A $4 < k \leq 5$

B $4 < k < \dfrac{3757}{729}$

C $4 \leq k \leq \dfrac{3757}{729}$

D $4 < k \leq \dfrac{3757}{729}$

E $4 < k \leq 8$

2 For $k \geq 0$, let $M(k)$ be the largest number of distinct real solutions that the equation

$$x^4 + ax^2 + bx + 1 = 0$$

can have as a, b range over all real numbers satisfying $20a^2 + 9b^2 \leq k$.

What is the complete set of values of k for which $M(k) = 2$?

A $(65/3, 80]$

B $(65/3, 90]$

C $(22, 80]$

D $(20, 80]$

E $[65/3, 80)$

3 Fifteen lamps are arranged in a circle and labelled $0, 1, \ldots, 14$. All are initially off. Pressing switch i changes the states of lamps i, $i+1$ and $i+2$, with labels interpreted modulo 15.

A configuration is called reachable if some finite sequence of presses produces it. For a reachable configuration S, let $d(S)$ be the fewest presses needed to produce it.

Let D be the largest value of $d(S)$, and let H be the number of reachable configurations for which $d(S) = D$.

Which pair (D, H) is correct?

A $(8, 2000)$

B $(7, 500)$

C $(8, 1000)$

D $(7, 1000)$

E $(7, 4000)$

4 A line is called a *double bisector* of a triangle if it divides the triangle into two polygons which have both equal areas and equal perimeters. Each polygon's perimeter includes the part of the line inside the triangle. Distinct lines are counted without orientation.

For $C > 1$, let $T(C)$ be the statement:

> Every non-degenerate triangle whose longest side is less than C times its shortest side has exactly three double bisectors.

Which condition on C is necessary and sufficient for $T(C)$ to be true?

A $1 < C \leq \dfrac{1+\sqrt{5}}{2}$

B $1 < C < \dfrac{1+\sqrt{2}}{2}$

C $1 < C \leq \dfrac{1+\sqrt{2}}{2}$

D $1 < C \leq \sqrt{\dfrac{3}{2}}$

E $1 < C \leq \sqrt{2}$

5 For real k, let $N(k)$ be the number of ordered triples (x, y, z) of pairwise distinct positive real numbers, none equal to 1, for which

$$xyz = 1, \qquad x + y + z = 6,$$

$$(\log_x y)^2 + (\log_y z)^2 + (\log_z x)^2 = k.$$

What is the complete set of values of k for which $N(k) = 6$?

A $[5, 21/4]$

B $\{5\}$

C $\{21/4\}$

D $\{9/2, 5\}$

E $\{5, 21/4\}$

6 For a real number a, define

$$F_a(x) = \frac{x^3 + ax}{x^2 + 1} \qquad (x > 0).$$

Call a admissible if both of the following hold:

- F_a is not one-to-one on the positive real numbers;
- for every pair of distinct positive reals x, y with $F_a(x) = F_a(y)$, at least one of the inequalities

  $$|x - y| < 4, \qquad \frac{\max\{x, y\}}{\min\{x, y\}} < 5$$

  holds.

Which one of the following is exactly the set of admissible values of a?

A $[-16, 14)$

B $[-16, 0) \cup (9, 5 + 4\sqrt{5})$

C $(-16, 0) \cup (9, 14)$

D $[-16, 0) \cup (9, 14)$

E $[-16, 0) \cup (9, 14]$

7 For each choice of integers r, s, t, u satisfying

$$0 \leq r, s, t, u \leq 12,$$

put $N = 2^r 3^s 5^t 7^u$. Let $D(N)$ be the largest positive integer with the following property: for all integers a, b,

$$N \mid a^2 + ab + b^2 \quad \Longrightarrow \quad D(N) \mid ab(a+b).$$

How many of these distinct values of N have $D(N)$ a perfect square?

A 3822

B 4116

C 4459

D 3276

E 3528

8 A circle meets the parabola $y = x^2$ at four distinct points A, B, C, D, in increasing order of their x-coordinates $a < b < c < d$. Suppose

$$a^2 + b^2 + c^2 + d^2 = 6$$

and the diagonals AC and BD are perpendicular. Exactly one of the points

$$X = \left(0, \frac{1}{4}\right), \qquad Y = \left(0, \frac{9}{4}\right)$$

lies strictly inside the quadrilateral $ABCD$. A point on its boundary does not count as strictly inside.

Which set is exactly the set of possible areas of $ABCD$?

A $[3, 9 - 3\sqrt{3}] \cup [1 + \sqrt{11}, 5]$

B $(3, 9 - 3\sqrt{3}] \cup [1 + \sqrt{11}, 5]$

C $[9 - 3\sqrt{3}, 1 + \sqrt{11}]$

D $(3, 5] \setminus \{9 - 3\sqrt{3}, 1 + \sqrt{11}\}$

E $(3, 9 - 3\sqrt{3}) \cup (1 + \sqrt{11}, 5]$

9 Let $P(x) = x^3 - 3x$. For $t \in \{1, 2, 3\}$ and a non-negative integer d, consider the following claim about every real polynomial G of degree at most d, including the zero polynomial:

If $G'(1) = 0$ and G takes the same value at all real roots of $P(x) = u$, separately for each $u \in \{-2, 0, t\}$, then

$$P(x) = P(y) \implies G(x) = G(y) \qquad \text{for every real } x, y.$$

The common values for different u are allowed to differ. A level with just one real root imposes no restriction on G.

For each $t \in \{1, 2, 3\}$, let d_t be the greatest d for which this claim is true. What is (d_1, d_2, d_3)?

A $(10, 8, 7)$

B $(8, 6, 5)$

C $(9, 7, 6)$

D $(9, 8, 6)$

E $(9, 7, 8)$

10 An integer n is chosen uniformly at random from $1 \leq n < 2^{18}$. It is given that exactly 64 coefficients in the expansion of $(1+x)^n$ are odd, and that the coefficient of x^n in $(1+x)^{3n}$ is even.

Let $T(n)$ be the number of odd coefficients in the expansion of $(1+x+x^2)^n$.

What is the probability that $T(n)$ is a perfect square?

A $\dfrac{7}{54}$

B $\dfrac{7}{27}$

C $\dfrac{39}{119}$

D $\dfrac{11}{54}$

E $\dfrac{13}{54}$

11 A real number c is called admissible if the following equivalence is true for every monic polynomial p of degree 4 whose derivative has three distinct real roots:

$$p \text{ has four distinct real roots}$$
$$\Longleftrightarrow$$
$$(p'(x))^2 > c\,p(x)p''(x) \text{ for every real } x.$$

Which one of the following is the complete set of admissible values of c?

A $1 \leq c \leq \frac{4}{3}$

B $0 < c < 1$

C $0 < c \leq 1$

D $0 < c < \frac{4}{3}$

E $0 < c \leq \frac{4}{3}$

12 An equilateral triangle ABC has circumcentre O and circumradius 1. For $r > 0$, let $\mathcal{R}_r$ be the set of points P on the circle $OP = r$ for which the three numbers

$$PA^2, \quad PB^2, \quad PC^2$$

are the side lengths of a non-degenerate right-angled triangle. Let $\Delta(P)$ be the area of that right-angled triangle.

Consider only radii r for which $\mathcal{R}_r$ contains exactly six distinct points, and define

$$E(r) = \frac{\sum\limits_{P \in \mathcal{R}_r} \Delta(P)}{(1 + r^2)^2}.$$

Which interval is exactly the set of possible values of $E(r)$?

A $\left(0, \dfrac{81 - 54\sqrt{2}}{2}\right]$

B $\left(0, \dfrac{81 - 54\sqrt{2}}{4}\right)$

C $\left(0, \dfrac{81 - 54\sqrt{2}}{2}\right)$

D $\left[0, \dfrac{81 - 54\sqrt{2}}{2}\right)$

E $\left[0, \dfrac{81 - 54\sqrt{2}}{2}\right]$

13 Let T be the set of all possible values of $f(2) - \lambda$, where λ is a non-zero real number and $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f\Big(x^2 + f(y)\Big) = \lambda y + (\lambda - 2) f(x)^2 \qquad \text{for every real } x, y.$$

No continuity or differentiability is assumed.

What is the sum of the squares of the distinct elements of T?

A 12

B 16

C 4

D 8

E 10

14 For each real number h, let $\mathcal{L}_h$ be the set of distinct straight lines which pass through $(h, -1)$ and are tangent to

$$y = x^4 - 2x^2.$$

A line tangent at two different points is counted only once. If $\mathcal{L}_h$ has exactly three elements, let $S(h)$ be the sum of the squares of their gradients.

For a real number C, consider the statement

there are exactly four real numbers h for which $|\mathcal{L}_h| = 3$ and $S(h) = C$.

Which condition is necessary and sufficient for this statement?

A $\dfrac{64(2-\sqrt{2})}{27} < C \leq \dfrac{128}{27}, \quad C \neq \dfrac{1024}{729}$

B $\dfrac{64(2-\sqrt{2})}{27} < C < \dfrac{128}{27}$

C $\dfrac{64(2-\sqrt{2})}{27} < C < \dfrac{128}{27}, \quad C \neq \dfrac{1024}{729}$

D $\dfrac{64(2-\sqrt{2})}{27} < C < \dfrac{128}{27}, \quad C \neq \dfrac{512}{729}$

E $\dfrac{64(2-\sqrt{2})}{27} \leq C < \dfrac{128}{27}, \quad C \neq \dfrac{1024}{729}$

15 How many monic polynomials p of degree 6 with integer coefficients satisfy all of the following conditions?

- The coefficient of x^5 is 0 and the coefficient of x^4 is -30.
- For infinitely many positive integers n, the number $p(n)$ is the square of an integer.
- The polynomial p has exactly three distinct real zeros, none of which is an integer.

A 34

B 36

C 38

D 40

E 32

16 Let

$$R(x) = \frac{x^3 - 3x}{3x^2 - 1}.$$

A real number a is called admissible if the sequence $x_1 = a$, $x_{n+1} = R(x_n)$ is defined and positive for every positive integer n, and has least positive period 12.

In addition, among $x_1, \ldots, x_{12}$ exactly six terms exceed $\sqrt{3}$, and exactly two of the twelve cyclic adjacent pairs

$$(x_1, x_2), (x_2, x_3), \ldots, (x_{11}, x_{12}), (x_{12}, x_1)$$

have both terms exceeding $\sqrt{3}$.

How many admissible values of a are there?

You may use the identity

$$R\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{\cos 3\theta}{\sin 3\theta}$$

whenever both sides are defined.

A 276

B 288

C 300

D 600

E 144

17 Ten coins are tossed together. For every choice of three of the coins, each of the eight possible patterns of heads and tails for those three coins has probability $\frac{1}{8}$. No other independence is assumed.

Let X be the number of heads. It is also given that

$$\Pr(X \geq 8) = \frac{3}{20}.$$

What is the set of all possible values of $\Pr(X \leq 2)$?

A $\left[\frac{1}{45}, \frac{7}{100}\right]$

B $\left(\frac{1}{40}, \frac{7}{100}\right)$

C $\left[\frac{1}{32}, \frac{5}{36}\right]$

D $\left[\frac{1}{40}, \frac{1}{8}\right]$

E $\left[\frac{1}{40}, \frac{7}{100}\right]$

18 Let $\mathcal{P}$ be the set of real polynomials p of degree at most 6 such that

$$p(x) \geq 0 \quad (-1 \leq x \leq 1), \qquad \int_{-1}^{1} p(x)\,\mathrm{d}x = 1.$$

Write $a_6(p)$ for the coefficient of x^6, taking $a_6(p) = 0$ when the degree is less than 6. Define

$$A = \max_{p \in \mathcal{P}} p(0), \qquad B = \max_{\substack{p \in \mathcal{P} \\ a_6(p) \geq 0}} p(0).$$

Let N be the number of polynomials attaining the second maximum which are non-negative for every real x. Which ordered triple is (A, B, N)?

A $\left(\frac{45}{32}, \frac{9}{8}, 1\right)$

B $\left(\frac{45}{32}, \frac{9}{8}, \infty\right)$

C $\left(\frac{45}{32}, \frac{5}{4}, 0\right)$

D $\left(\frac{45}{32}, \frac{45}{32}, 1\right)$

E $\left(\frac{9}{8}, \frac{9}{8}, 1\right)$

19 A set S consists of six distinct real numbers. For every $r \in S$, suppose that

$$\sum_{\substack{s \in S \\ s \neq r}} \frac{1}{r-s} = r^3.$$

What is the value of $\sum_{r \in S} r^2$?

A $\sqrt{45 + 12\sqrt{5}}$

B $\sqrt{30 - 12\sqrt{5}}$

C $\sqrt{15 + 6\sqrt{5}}$

D $\sqrt{30 + 6\sqrt{5}}$

E $\sqrt{30 + 12\sqrt{5}}$

20 For a real number x, $\lfloor x \rfloor$ is the greatest integer not exceeding x. Let $0 < \alpha < 1$ and $-1 \leq t \leq 0$. For each positive integer q, consider the two sequences, both indexed by *all* integers n:

$$u_n = \left\lfloor \frac{(2q+1)n}{q} + \alpha \right\rfloor, \qquad v_n = \left\lfloor \frac{(2q+1)n + t - q\alpha}{q+1} \right\rfloor.$$

Call (α, t) *perfect* if, for **every** positive integer q, every integer occurs exactly once in the two sequences combined.

How many values of α make **exactly one** of

$$(\alpha, -1/8) \qquad \text{and} \qquad (\alpha, -1/12)$$

perfect?

A Infinitely many

B 18

C 20

D 24

E 28