1 Pairwise distinct positive real numbers x, y, z satisfy

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1, \qquad xyz=32.$$

Let

$$E=(x-y)^2+(y-z)^2+(z-x)^2.$$

Which one of the following describes all the possible values of E?

A $8 \leq E \leq 60-20\sqrt{5}$

B $8 < E \leq 60-20\sqrt{5}$

C $8 \leq E < 60-20\sqrt{5}$

D $8 < E < 60-20\sqrt{5}$

E $0 < E < 60-20\sqrt{5}$

2 A sequence of positive real numbers satisfies

$$a_{n+2} = \frac{1 + a_{n+1}}{a_n} \qquad (n \geq 1).$$

For which real values of t are there exactly four ordered pairs (a_1, a_2) such that

$$a_1 a_2 a_3 a_4 a_5 = 12 \qquad \text{and} \qquad a_{2026} + a_{2028} = t?$$

A $4 < t < 11 - 4\sqrt{3}$

B $4 < t \leq 11 - 4\sqrt{3}$

C $\sqrt{33} - 3 < t < 11 - 4\sqrt{3}$

D $4 \leq t < 11 - 4\sqrt{3}$

E $\sqrt{33} - 3 < t < 4$

3 A rectangle has area 1. Four circles of positive radii lie inside it, one at each vertex, with each circle tangent to the two sides meeting at that vertex.

Each circle is externally tangent to the two circles at adjacent vertices. The interiors of all four circles are pairwise disjoint.

Write $\lambda = \sqrt{1 - 1/\sqrt{2}}$. What is the greatest possible sum of the areas of the four circles?

A $4\pi\lambda^2(1-\lambda)^2$

B $\pi\Big(\lambda^2 + (1-\lambda)^2\Big)^2$

C $4\pi\lambda^4$

D $\dfrac{\pi}{4}$

E $2\pi\Big(\lambda^4 + (1-\lambda)^4\Big)$

4 For how many real values of a does the equation

$$\left|\log_2\left(x^2 - ax + \frac{5}{2}\right)\right| = |\log_2 x|$$

have exactly three distinct real solutions which, in increasing order, form an arithmetic progression?

A 3

B 2

C 1

D 4

E 5

5 Four distinct integers are chosen uniformly at random from $\{1, 2, \ldots, 12\}$. It is given that their sum is divisible by 3.

Let P be the monic polynomial of degree 4 whose roots are the four chosen integers. What is the probability that both local minimum points of the graph $y = P(x)$ have y-coordinate -9?

A $\frac{2}{55}$

B $\frac{8}{165}$

C $\frac{4}{165}$

D $\frac{2}{165}$

E $\frac{4}{495}$

6 For a real number k, the equation

$$64\cos^4 x - 100\cos^2 x - 9 = 64k\cos x$$

has exactly six distinct solutions $x_1, \ldots, x_6$ in $0 \leq x < 2\pi$.

As k varies over all values for which this is true, what is the least possible value of

$$\sum_{j=1}^{6}\left(9\cos^2 x_j - 2\cos^4 x_j\right)?$$

A $\dfrac{441}{32}$

B $\dfrac{455}{32}$

C $\dfrac{455}{64}$

D $\dfrac{1103}{32}$

E $\dfrac{697}{32}$

7 For real numbers a and b, let

$$P(x) = x^3 - 3x + a, \qquad Q(x) = x^2 - bx - 1.$$

What is the greatest positive number C with the following property?

For every pair (a, b) satisfying $a^2 + 4b^2 < C$, the two polynomials have five distinct real roots which, in increasing order, are a root of P, a root of Q, a root of P, a root of Q and a root of P.

A 5

B 2

C 6

D 4

E 3

8 A function f maps the integers to the integers and satisfies

$$f(f(f(n))) = n + 18, \qquad f(n + 12) = f(n) + 12$$

for every integer n. In addition,

$$f(0), f(1), f(2), f(3), f(4), f(5) > 0.$$

How many different functions f satisfy all these conditions?

A 5121

B 1681

C 2721

D 2720

E 4401

9 Call a line admissible if it intersects $y = x^3 - 3x$ at three distinct points and the two finite regions between the line and the curve have areas in the ratio $32 : 5$.

For some real number h, exactly three distinct admissible lines pass through $(h, 0)$. Let S be the sum of their gradients.

Which one of the following describes all the possible values of S?

A $S > \frac{45}{4}$

B $S \geq \frac{45}{4}$

C $S > 9$

D $S > \frac{81}{4}$

E $S \geq 9$

10 Let a be real and $f(x) = x^2 - ax$. Define

$$f_1(x) = f(x), \qquad f_{n+1}(x) = f\Big(f_n(x)\Big) \quad (n \geq 1).$$

Suppose that $f_2(x) = 0$ has four distinct real roots which, in increasing order, form an arithmetic progression.

Which set contains exactly the possible numbers of distinct real roots of $f_5(x) = 0$?

A $\{12, 32\}$

B $\{8, 32\}$

C $\{10, 16\}$

D $\{16, 32\}$

E $\{10, 32\}$

11 The numbers $X_1, \ldots, X_{12}$ consist of six 1s and six -1s, with every choice of the six positive positions equally likely. Set $X_{13} = X_1$, $X_{14} = X_2$, and

$$Y_i = X_i X_{i+1} X_{i+2} \qquad (1 \leq i \leq 12).$$

What is the probability that exactly ten of the twelve numbers Y_i are positive?

A $\dfrac{1}{77}$

B $\dfrac{6}{77}$

C $\dfrac{3}{77}$

D $\dfrac{1}{154}$

E $\dfrac{2}{77}$

12 For $a > 1$ and real x, define

$$F_a(x) = \frac{\sin x}{a + \sin x} + \frac{\cos x}{a + \cos x}, \qquad H_a(x) = F_a(x)F_a(x + \pi).$$

For which values of a is the minimum of $H_a(x)$ on $0 \leq x < 2\pi$ attained at exactly four distinct values of x?

A $1 < a < \sqrt{\frac{3}{2}}$

B $a > \sqrt{\frac{5}{2}}$

C $1 < a < \sqrt{\frac{5}{2}}$

D $1 < a \leq \sqrt{\frac{5}{2}}$

E $\sqrt{\frac{3}{2}} < a < \sqrt{\frac{5}{2}}$

13 For a positive real number c, let C be the curve

$$x + y + xy = c(x^2 + y^2).$$

For every t with $0 \leq t \leq 1$, the line $x = t$ meets C at a point with $0 \leq y \leq 1$.

What is the greatest possible value of $(x - y)^2$ at a point of C satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 1$?

A $\dfrac{4}{5}$

B $\dfrac{3 - \sqrt{5}}{2}$

C $2 - \dfrac{4\sqrt{5}}{5}$

D $\dfrac{1}{2}$

E $4 - \dfrac{8\sqrt{5}}{5}$

14 For real numbers a, b, let

$$P(x) = (x^2 + 1)^3 + ax(x^2 + 1)^2 + bx^3.$$

Suppose that $P(1) = 0$ and $P(x) \geq 0$ for every real x. Also suppose that $P(x) = P(-x)$ has exactly five distinct real solutions.

What is the least possible sum of the squares of these five solutions?

A 6

B $\frac{28}{3}$

C 8

D $\frac{20}{3}$

E 12

15 Let $\mathcal{S}$ be the set of all points N with the following property: three distinct normals to $y = x^2$ pass through N, and their three corresponding points on the parabola form a right-angled triangle of area 1.

If O is the origin, what is

$$\sum_{N \in \mathcal{S}} ON^2 \, ?$$

A $8\sqrt{6}$

B $\dfrac{9}{4} + 8\sqrt{6}$

C $6 + 8\sqrt{6}$

D $\dfrac{9}{2} + 8\sqrt{6}$

E $\dfrac{9}{4} + 4\sqrt{6}$

16 Eight cards numbered $1, 2, \ldots, 8$ are placed uniformly at random around a circle. Let T be the sum of the absolute differences between the numbers on neighbouring cards, including the difference across the last gap.

Given that $T = 18$, what is the probability that cards 1 and 8 are next to each other?

A $\dfrac{3}{22}$

B $\dfrac{1}{11}$

C $\dfrac{1}{8}$

D $\dfrac{9}{176}$

E $\dfrac{9}{88}$

17 A polynomial p has degree 6 and leading coefficient 1. For every real t,

$$\int_{t-1}^{t+1} p(x)\,dx \geq 0,$$

with equality if and only if $t = -2$ or $t = 2$.

Which one of the following describes all the possible values of $p(1) - p(0)$?

A $p(1) - p(0) < 55$

B $p(1) - p(0) < \dfrac{499}{9}$

C $p(1) - p(0) \leq 27$

D $p(1) - p(0) \leq \dfrac{499}{9}$

E $p(1) - p(0) \leq 55$

18 For every choice of $a < 0 < b$, let $P = (a, a^2)$ and $Q = (b, b^2)$. The tangents to $y = x^2$ at P and Q meet at R, and $C_{a,b}$ is the circle through P, Q, R.

A point $X = (u, v)$ must lie strictly inside every circle $C_{a,b}$.

What is the greatest possible value of u^2?

A $\dfrac{51\sqrt{17} - 107}{2048}$

B $\dfrac{3}{64}$

C $\dfrac{1}{16}$

D $\dfrac{17\sqrt{17} - 35}{512}$

E $\dfrac{51\sqrt{17} - 107}{1024}$

19 A monic polynomial of degree 12 has twelve distinct real roots, none equal to 0 or 1. Replacing every root r by $1 - r$ leaves the set of roots unchanged; so does replacing every root r by $1/r$.

The sum of the squares of the twelve roots is 30. Let T be the sum of their fourth powers.

Which one of the following describes all the possible values of T?

A $174 < T \leq \dfrac{777}{4}$

B $174 \leq T \leq \dfrac{777}{4}$

C $174 < T < \dfrac{777}{4}$

D $T > 174$

E $174 \leq T < \dfrac{777}{4}$

20 Seven coins are tossed independently. Their seven probabilities of heads are all distinct and lie strictly between 0 and 1.

For each $k \in \{0, 1, \ldots, 7\}$, the probability of exactly k heads equals the probability of exactly $7 - k$ heads. One coin has probability of heads $\frac{1}{3}$, and the probability that all seven show heads is $\frac{1}{576}$.

Which of the following can be the probability of exactly one head?

A $\dfrac{41}{1296}$

B $\dfrac{13}{384}$

C $\dfrac{1}{40}$

D $\dfrac{1}{32}$

E $\dfrac{31}{1152}$