1. D

A counterexample must make the premise true and the conclusion false. In option D, both 1.6 and 1.8 round to 2, but their sum is 3.4, which rounds to 3, not 4.

For A, the number 1.4 does not round to 2, so the premise is false. In B and C, both numbers round to 2 and their sums, 3.8 and 4.1, round to 4. In E, the number 2.6 rounds to 3, so the premise is false. Thus only D is a counterexample.

2. A

Adding the equations gives $5(a+b) = 15$, so $a+b = 3$. Subtracting the second equation from the first gives $a-b = -1$. Hence

$$a^2 - b^2 = (a+b)(a-b) = 3(-1) = \boxed{-3}.$$

3. D

Rewrite the graph's equation as

$$y = 2 + \frac{7}{x-3}.$$

The fraction cannot be zero, so $y = 2$ has no common point with the graph.

For any $k \neq 2$, the equation $2 + 7/(x-3) = k$ gives

$$x = 3 + \frac{7}{k-2},$$

which is real and is not 3. Thus every other horizontal line has a common point, and $k = \boxed{2}$.

4. C

An implication is false exactly when its premise is true and its conclusion is false. Here that happens when

$$x^2 < 9 \quad \text{and} \quad x \leq 1,$$

which is precisely $-3 < x \leq 1$.

Therefore S is true for all remaining real values:

$$\boxed{x \leq -3 \quad \text{or} \quad x > 1}.$$

In particular, $x = -3$ is included because the premise is false, whereas $x = 1$ is excluded because the premise is true and the conclusion is false.

5. A

Put $t = 3^x$, so $t > 0$. The equation becomes

$$t^2 - 8t + k = 0.$$

A negative x corresponds to $0 < t < 1$, and a positive x corresponds to $t > 1$.

For the two roots to lie on opposite sides of 1, the upward-opening quadratic must be negative at $t = 1$. Thus $k - 7 < 0$.

Conversely, if $0 < k < 7$, the roots are $4 \pm \sqrt{16 - k}$. Since $3 < \sqrt{16 - k} < 4$, the smaller root lies strictly between 0 and 1, and the larger root exceeds 1. Both give valid real values of x, of opposite signs.

At $k = 7$, the smaller root is 1, giving $x = 0$, so this boundary is excluded. The possible positive integers are $1, 2, 3, 4, 5, 6$: there are $\boxed{6}$.

6. B

Suppose Q holds. Taking $x = 1$ and $x = -1$ gives

$$2(a + b) \geq 0, \qquad 2(a - b) \geq 0.$$

Thus $a \geq b$ and $a \geq -b$, which together are $a \geq |b|$. Therefore P is necessary for Q.

Conversely, if P holds, both $a + b$ and $a - b$ are nonnegative. For every real x,

$$a(x^2 + 1) + 2bx = \frac{a + b}{2}(x + 1)^2 + \frac{a - b}{2}(x - 1)^2 \geq 0.$$

Thus P is sufficient as well. The argument also includes $a = b = 0$; no division by a is used.

Hence P is both necessary and sufficient for Q.

7. C

Let the radius be r. Because the arc is tangent to both legs, the perpendicular distance from O to each of AB and AC is r.

Join A to O. As O lies on BC, triangles ABO and ACO together form triangle ABC. Equating their areas gives

$$\frac{1}{2} \cdot 8r + \frac{1}{2} \cdot 6r = \frac{1}{2} \cdot 8 \cdot 6.$$

Hence $7r = 24$, so the radius is

$$\boxed{\frac{24}{7}}.$$

8. C

Let the original mean be m and the removed number be r. If Q holds, the new mean is also m, so

$$\frac{5m - r}{4} = m.$$

Hence $r = m$, which is P. Thus P is necessary for Q.

It is not sufficient. The list

$$1,\ 2,\ 3,\ 4,\ 10$$

has mean 4 and median 3. Removing 4 preserves the mean, but the remaining list $1, 2, 3, 10$ has median

$$\frac{2+3}{2} = \frac{5}{2}.$$

Thus P holds while Q fails. The answer is C.

9. D

The first few terms are

$$3,\ 4,\ 2,\ 5,\ 1,\ 6,\ 0,\ 7,\ -1,\ 8, \ldots.$$

Two consecutive increments give

$$u_{n+2} - u_n = (-1)^{n+1} n + (-1)^{n+2}(n+1) = (-1)^n.$$

Thus the odd-indexed terms decrease by 1 and the even-indexed terms increase by 1:

$$u_{2k-1} = 4 - k, \qquad u_{2k} = k + 3.$$

Among the first 20 terms, the positive odd-indexed terms are $3, 2, 1$, with sum 6. All ten even-indexed terms are positive; they run from 4 to 13 and have sum

$$\frac{10(4+13)}{2} = 85.$$

The required sum is $6 + 85 = \boxed{91}$. The term $u_7 = 0$ is not positive.

10. F

Every line is valid. The identity

$$s^2 - 4ab = (a - b)^2 \geq 0$$

justifies line I. The given equation says $ab = 3 - s$, so substitution and rearrangement give line II.

The factorisation in line III is correct. The product is nonnegative exactly when $s \leq -6$ or $s \geq 2$. Positivity of a and b gives $s > 0$, excluding the first possibility and justifying line IV. Finally, subtracting $s \geq 2$ from 3 gives line V.

Equality is possible: $a = b = 1$ satisfies the original equation and has $ab = 1$. The answer is F.

11. B

All 6^3 ordered outcomes are equally likely. There are two disjoint ways for the largest score to occur at least twice.

- All three scores are equal. There are 6 outcomes, and the sum is even in 3 of them.
- The scores are m, m, r, where $1 \leq r < m \leq 6$. For each m there are $m-1$ choices for r and 3 positions for it. This gives
$$3(0+1+2+3+4+5) = 45$$
outcomes.

In the second case the sum is $2m + r$, which is even exactly when r is even. For $m = 1, 2, 3, 4, 5, 6$, the numbers of permitted even values of r are $0, 0, 1, 1, 2, 2$. Hence there are

$$3(0+0+1+1+2+2) = 18$$

favourable outcomes in this case.

The required conditional probability is therefore

$$\frac{3+18}{6+45} = \boxed{\frac{7}{17}}.$$

12. D

The statement fails when there is at least one positive level k for which no suitable pair exists.

For this fixed k, every pair of distinct real numbers must fail at least one of the three requirements

$$f(x) = k, \qquad f(y) = k, \qquad x + y < 0.$$

Their simultaneous failure is not required: at least one of

$$f(x) \neq k, \qquad f(y) \neq k, \qquad x + y \geq 0$$

must hold. This is option D.

If the level k has fewer than two distinct inputs, it already makes the original assertion fail. The disjunction in D correctly includes this possibility. Equality $x + y = 0$ also fails the original strict inequality.

13. D

Simplify before differentiating:

$$s(t) = t - 5 + 4t^{-1}, \qquad s'(t) = 1 - 4t^{-2}.$$

For $1 \leq t < 2$ the velocity is negative, and for $2 < t \leq 4$ it is positive. Thus the particle reverses direction once, at $t = 2$.

Its positions at the endpoints and the reversal are

$$s(1) = 0, \qquad s(2) = -1, \qquad s(4) = 0.$$

It travels 1 metre in each direction, a total distance of 2 metres. The elapsed time is $4 - 1 = 3$ seconds, so its average speed is

$$\boxed{\frac{2}{3}} \text{ metres per second.}$$

The zero net displacement would instead give average velocity 0.

14. A

Suppose P holds. Then $(a-b)^2 = 1$.

If a positive integer d divides both $a+b$ and a^2+b^2, it divides $(a+b)^2$ and $2(a^2+b^2)$. Therefore it divides their difference,

$$2(a^2+b^2) - (a+b)^2 = (a-b)^2 = 1.$$

Thus $d = 1$, and the highest common factor is 1. This proves that P is sufficient for Q.

It is not necessary. Take $a = 1$ and $b = 4$. These integers are not consecutive, but

$$a+b = 5, \qquad a^2+b^2 = 17,$$

whose highest common factor is 1. Thus Q holds while P fails. The answer is A.

15. C

The whole triangle has area $\frac{1}{2} \cdot 6 \cdot 6 = 18$.

The line $x + y = 4$ passes through $(2, 2)$. It cuts off a right triangle with perpendicular sides of length 4, whose area is

$$\frac{1}{2} \cdot 4^2 = 8.$$

The other region has area $18 - 8 = 10$. Thus P can hold while Q fails: P is not sufficient.

Conversely, the line $x + y = 3\sqrt{2}$ cuts off a right triangle of area

$$\frac{1}{2}(3\sqrt{2})^2 = 9.$$

The other region also has area 9. However, $2 + 2 = 4 \neq 3\sqrt{2}$, so this line does not pass through $(2, 2)$. Thus Q can hold while P fails: P is not necessary.

Both lines pass through the interior, since their positive axis intercepts are less than 6. Therefore P is neither necessary nor sufficient for Q.

16. B

Put $s = \sin\theta$ and $q(s) = s^2 - (k+1)s + k^2/4$. Its discriminant is $2k+1$.

First let $k \neq 0$. Whenever there are real roots, $k \geq -1/2$, their sum $k+1$ is positive and their product $k^2/4$ is positive. Hence both roots are positive. Each root strictly between 0 and 1 gives two angles, while $s = 1$ gives only one. Therefore four distinct angles require two distinct roots strictly between 0 and 1.

For this to happen, the discriminant must be positive, the vertex must lie below $s = 1$, and $q(1)$ must be positive. Together with the positive roots already established, these conditions are also sufficient. They give

$$k > -\frac{1}{2}, \qquad \frac{k+1}{2} < 1, \qquad q(1) = \frac{k(k-4)}{4} > 0.$$

Their intersection is $-1/2 < k < 0$.

When $k = 0$, the equation is $\sin\theta(\sin\theta - 1) = 0$. Its four distinct solutions are

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad 2\pi.$$

Thus the answer is $\boxed{-\frac{1}{2} < k \leq 0}$. At $k = -1/2$ the quadratic has a repeated root and only two angles, so that endpoint is excluded.

17. D

At least three distinct selected points lie on ℓ. Two of them lie on one closed ray from H: a point at H, if present, is allowed. Name these A, B with $HA < HB$. Then

$$0 < AB = HB - HA \leq HB < PB,$$

where the last inequality follows from $PB^2 = PH^2 + HB^2$ and $PH > 0$.

Let d be the perpendicular distance from A to the line PB. Calculating the area of triangle PAB using two bases gives

$$\frac{1}{2}AB \cdot PH = \frac{1}{2}PB \cdot d, \qquad d = PH\frac{AB}{PB} < PH.$$

This is a positive admissible distance: P, A, B are selected points and are not collinear. It contradicts the choice of PH. This is exactly option D.

The other constructions fail for specific reasons. Take $H = (0, 0)$ and $P = (0, 1)$, with A, B on the horizontal axis:

- A: $A = (-1, 0)$, $B = (1, 0)$ give $AB = 2 > PB = \sqrt{2}$; opposite-side points need not exist either.
- B: $A = (1, 0)$, $B = (3, 0)$ give $AB = 2 > PA = \sqrt{2}$.
- C: the area formula uses AB, not HA. For $A = (1, 0)$, $B = (3, 0)$ these are 2 and 1.
- E: points at $-1, 2, 3$ on the axis make $A = (-1, 0)$ and $B = (3, 0)$, so $AB = 4 > PB = \sqrt{10}$.

These examples test the proposed local geometric inferences. The least positive distance exists because there are finitely many admissible point–line choices and at least one exists by noncollinearity.

18. E

Write the original numbers of entries as $3n$ and $2n$. Moving an entry does not change the combined total of all the values. Hence

$$3n \cdot 20 + 2n \cdot 12.5 = (3n - 1) \cdot 21 + (2n + 1) \cdot 13.$$

Simplifying gives $85n = 89n - 8$, so $n = 2$. The combined number of entries is therefore

$$3n + 2n = \boxed{10}.$$

The data are attainable. The first list can be $15, 21, 21, 21, 21, 21$, of mean 20, and the second list can contain four entries of 12.5. Moving the entry 15 leaves five entries of mean 21 and produces a second-list mean of $(50 + 15)/5 = 13$.

19. E

Substituting $d = -(a+b+c)$ into the cube-sum equation and factorising gives

$$0 = a^3 + b^3 + c^3 - (a+b+c)^3 = -3(a+b)(a+c)(b+c).$$

Thus some pair of the numbers has sum 0. The remaining pair also has sum 0, since the total sum is 0.

Because the four numbers are distinct, neither opposite pair is $0, 0$, and their positive magnitudes are different. Write the numbers in increasing order as

$$-u, \quad -v, \quad v, \quad u, \qquad 0 < v < u.$$

Their median is $(-v+v)/2 = 0$, and their product is $u^2v^2 > 0$. Statements I and II must therefore be true.

Statement III need not be true. The list $-4, -1, 1, 4$ satisfies both given equations and has four distinct entries, but its consecutive gaps are $3, 2, 3$. It is not an arithmetic progression.

Hence only I and II must be true.

20. C

Every point of the square lies in at least one of the four closed disks. To see this, divide the square into four smaller squares by its centre-lines. Each smaller square has side $1/\sqrt{2}$ and diagonal 1, so every point in it is at distance at most 1 from its corresponding original vertex.

Opposite vertices are distance 2 apart. Their radius-1 disks touch only at the square's centre. Any three of the four vertices include an opposite pair, so no point other than the centre can belong to three or four disks. Boundaries have zero area.

Let X and Y be the areas covered exactly once and exactly twice. The square has area 2, so

$$X + Y = 2.$$

Inside the square each disk occupies a quarter-circle of area $\pi/4$. Adding these four areas counts the once-covered region once and the twice-covered region twice. Hence

$$X + 2Y = \pi.$$

Subtracting yields $Y = \pi - 2$, and therefore

$$X = 2 - (\pi - 2) = \boxed{4 - \pi}.$$

This avoids calculating the four individual curved overlap regions.