1. C

A translation by 1 unit to the right replaces x by $x-1$. The new equation is therefore

$$y = (x-1)^2 + p(x-1) + q = x^2 + (p-2)x + (1-p+q).$$

Comparing coefficients with $y = x^2 + qx + p$ gives

$$q = p - 2, \qquad 1 - p + q = p.$$

Substitution gives $1 - p + (p-2) = p$, so $p = \boxed{-1}$.

The corresponding value is $q = -3$: translating $y = x^2 - x - 3$ gives $y = x^2 - 3x - 1$, as required.

2. C

Combining the integrals and applying the Fundamental Theorem of Calculus gives

$$f(6) - f(1) = \int_1^6 f'(x)\,\mathrm{d}x = 8 - 3 = 5.$$

The horizontal separation of the two points is $6 - 1 = 5$. Hence the gradient of their joining line is

$$\frac{f(6) - f(1)}{6 - 1} = \frac{5}{5} = \boxed{1}.$$

3. D

Write $m = u_5$ and let d be the common difference. The two given products are symmetric about the fifth term, so

$$(m - 3d)(m + 3d) = 40, \qquad (m - d)(m + d) = 48.$$

Thus

$$m^2 - 9d^2 = 40, \qquad m^2 - d^2 = 48.$$

Subtracting gives $8d^2 = 8$, so $d^2 = 1$ and $m^2 = 49$. Since all the terms are positive, $m = 7$.

The mean of the first nine terms of an arithmetic progression is its fifth term. Therefore their sum is

$$9m = \boxed{63}.$$

Both $d = 1$ and $d = -1$ are possible: they give the terms $3, 4, \ldots, 11$ in one order or the reverse order. Both give the same sum.

4. D

Let packet B contain m grams and cost p pence. Packet A then contains $\frac{5}{4}m$ grams and costs $\frac{11}{10}p$ pence.

Consequently, the ratio of the price per 100 grams in A to that in B is

$$\frac{11/10}{5/4} = \frac{22}{25}.$$

The required price is therefore

$$44 \times \frac{25}{22} = \boxed{50}$$

pence per 100 grams.

5. B

Put $AC = t$, so $AB = t + 2$ and $t > 0$. Since $\cos 120^\circ = -\frac{1}{2}$, the cosine rule gives

$$28 = t^2 + (t+2)^2 - 2t(t+2)\left(-\frac{1}{2}\right) = 3t^2 + 6t + 4.$$

Hence

$$3(t^2 + 2t - 8) = 3(t-2)(t+4) = 0.$$

The positive solution is $t = 2$, so $AB = t + 2 = \boxed{4}$.

The resulting sides are 2, 4 and $2\sqrt{7}$. Since $2\sqrt{7} < 6$, they form a genuine triangle; the cosine rule verifies the stated included angle.

6. E

Let the centre be $C = (h, 2h)$. Its distances from O and P are equal, so

$$h^2 + (2h)^2 = (4 - h)^2 + (2 - 2h)^2.$$

Expanding and cancelling gives $16h = 20$, hence

$$C = \left(\frac{5}{4}, \frac{5}{2}\right).$$

The perpendicular from the centre to chord OQ bisects that chord. Since OQ lies on the x-axis, its midpoint is $(\frac{5}{4}, 0)$, and therefore $Q = (\frac{5}{2}, 0)$.

The gradient of CQ is

$$\frac{0 - 5/2}{5/2 - 5/4} = -2.$$

The tangent is perpendicular to this radius, so it has gradient $\frac{1}{2}$. Through Q, its equation is

$$y = \frac{1}{2}\left(x - \frac{5}{2}\right) = \boxed{\frac{1}{2}x - \frac{5}{4}}.$$

7. B

Adding 1 to each equation suggests the substitutions

$$a = x + 1, \qquad b = y + 1, \qquad c = z + 1.$$

They give

$$ab = 6, \qquad bc = 12, \qquad ca = 8.$$

None of a, b, c is zero, so

$$a^2 = \frac{(ab)(ac)}{bc} = \frac{6 \cdot 8}{12} = 4.$$

If $a = 2$, then $b = 3$ and $c = 4$. Thus $(x, y, z) = (1, 2, 3)$ and $xyz = 6$.

If $a = -2$, then $b = -3$ and $c = -4$. Thus $(x, y, z) = (-3, -4, -5)$ and $xyz = -60$.

These exhaust the two possible values of a, and both triples satisfy the original equations. The required sum is

$$6 - 60 = \boxed{-54}.$$

8. D

Write the original numbers in ascending order as $x_1 \leq \cdots \leq x_{15}$. Their median is $x_8 = 7$.

The transformation $x \mapsto 20 - 2x$ reverses their order. Its middle entry is still the transformed original middle entry, so the new median is

$$20 - 2 \cdot 7 = 6.$$

The new maximum is $20 - 2x_1$ and the new minimum is $20 - 2x_{15}$. Thus its range is

$$(20 - 2x_1) - (20 - 2x_{15}) = 2(x_{15} - x_1) = 12.$$

The requested sum is $6 + 12 = \boxed{18}$.

9. B

Divide the second equation by the first equation's positive left-hand side:

$$\frac{2^{2x}+2^{2y}}{2^{x+y}}=\frac{40}{16},\qquad 2^{x-y}+2^{y-x}=\frac{5}{2}.$$

Put $u=2^{x-y}>0$. Then

$$u+\frac{1}{u}=\frac{5}{2},\qquad 2u^2-5u+2=(2u-1)(u-2)=0.$$

Hence $u=2$ or $u=1/2$, giving $x-y=1$ or $x-y=-1$. In both cases,

$$|x-y|=\boxed{1}.$$

Both cases are possible: $x+y=4$, so the pairs are $(x,y)=(5/2,3/2)$ and $(3/2,5/2)$, each satisfying the original equations.

10. D

Write $P = (t, t^2 + 3 + 1/t)$, where $t > 0$. The gradient of OP is its rise divided by its run:

$$m(t) = \frac{t^2 + 3 + 1/t}{t} = t + 3t^{-1} + t^{-2}.$$

Differentiate this sum of powers:

$$m'(t) = 1 - 3t^{-2} - 2t^{-3} = \frac{t^3 - 3t - 2}{t^3} = \frac{(t-2)(t+1)^2}{t^3}.$$

For $t > 0$, the factors $(t+1)^2$ and t^3 are positive. Thus m decreases for $0 < t < 2$ and increases for $t > 2$. Its unique global minimum is therefore attained at $t = 2$, with

$$m(2) = 2 + \frac{3}{2} + \frac{1}{4} = \boxed{\frac{15}{4}}.$$

The point $(2, 15/2)$ lies on the curve, so this value is attainable. The other derivative zero, $t = -1$, is outside the stated domain.

11. C

Let E be the event that the two remaining counters are in the same box. That box must be B or C, since A initially contains only one counter. Thus E occurs exactly when the counter in A and both counters from one of B and C have been removed.

Write X for the box, either B or C, from which both counters are removed. The three possible positions of A give the following cases. The factor 2 accounts for the two choices of X.

- Order AXX: after choosing A, only two boxes remain nonempty. The probability is
$$2\left(\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{1}{6}.$$
- Order XAX: all three boxes are nonempty after the first removal, but only two are nonempty after the second. The probability is
$$2\left(\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)=\frac{1}{9}.$$
- Order XXA: all three boxes are nonempty after the first removal; after the second, X is empty. The probability is
$$2\left(\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)=\frac{1}{9}.$$

These cases are disjoint and exhaust E. Therefore the required probability is

$$\frac{1/6}{1/6+1/9+1/9}=\boxed{\frac{3}{7}}.$$

12. A

Separate the two possible roles of AB.

If AB is a diagonal, its midpoint is $(2, 0)$. Since parallelogram diagonals bisect each other, the fourth vertex is $(2, -t)$. The diagonal lengths are 4 and $2t$, so

$$2t = 2 \cdot 4 \quad \text{or} \quad 4 = 2(2t).$$

This gives $t^2 = 16$ or $t^2 = 1$.

If AB is a side, the fourth vertex is either $(-2, t)$ or $(6, t)$. In either case, the squared diagonal lengths are

$$4 + t^2 \quad \text{and} \quad 36 + t^2.$$

The latter is larger. A length ratio of $2 : 1$ gives a squared-length ratio of $4 : 1$, so

$$36 + t^2 = 4(4 + t^2), \qquad t^2 = \frac{20}{3}.$$

All three positive values produce nondegenerate parallelograms. The two choices of fourth vertex in the side case give the same value of t^2, which is counted once. The required sum is

$$1 + 16 + \frac{20}{3} = \boxed{\frac{71}{3}}.$$

13. B

Let the common ratio be r. Then $u_5 = u_3 r^2$, so

$$r^2 = \frac{2}{8} = \frac{1}{4}.$$

The common ratio is negative, hence $r = -1/2$. Consequently

$$u_4 = 8\left(-\frac{1}{2}\right) = -4, \qquad u_6 = 2\left(-\frac{1}{2}\right) = -1.$$

Therefore $u_4 + u_6 = -4 - 1 = \boxed{-5}$.

14. D

Let the pole's height be $h > 0$. The right-angled triangles give

$$\tan\alpha = \frac{h}{AP}, \qquad \tan\beta = \frac{h}{BP}, \qquad \tan\gamma = \frac{h}{CP}.$$

Cancelling h from the given equation yields

$$5\left(\frac{1}{AP} + \frac{1}{CP}\right) = \frac{12}{BP}.$$

Reflection about B swaps A and C, leaving this equation and BP unchanged. We may therefore take P on the ray from B through C. Put $x = BP$, measured in metres; then $x > 0$ and $x \neq 3$.

If $0 < x < 3$**,** then $AP = x + 3$ and $CP = 3 - x$. Hence

$$\frac{30}{9 - x^2} = \frac{12}{x} \quad \Longrightarrow \quad 2x^2 + 5x - 18 = (2x + 9)(x - 2) = 0.$$

Only $x = 2$ lies in this case; the other root is $-9/2$.

If $x > 3$**,** then $AP = x + 3$ and $CP = x - 3$. Hence

$$\frac{10x}{x^2 - 9} = \frac{12}{x} \quad \Longrightarrow \quad x^2 = 54.$$

Thus $x = 3\sqrt{6}$, which is greater than 3.

All denominators are positive within their respective cases, so the steps are reversible. Each retained distance is possible for any positive pole height. The reflected positions give the same two distances, each counted once. Their sum is

$$\boxed{2 + 3\sqrt{6}}.$$

15. E

Since $x^2 + 1 > 0$ for every real x, the equation is equivalent to

$$(3 - k)x^2 + 4x + 1 - k = 0.$$

If $k = 3$, this is the linear equation $4x - 2 = 0$, which has exactly one real solution.

If $k \neq 3$, the equation is quadratic, so it has exactly one real solution if and only if its discriminant is zero:

$$16 - 4(3 - k)(1 - k) = 0, \qquad k^2 - 4k - 1 = 0.$$

The two resulting values are $k = 2-\sqrt{5}$ and $k = 2+\sqrt{5}$. Neither is 3, so both give genuine quadratic equations with one repeated real root.

Thus all three admissible values have been included, and their sum is

$$3 + (2 - \sqrt{5}) + (2 + \sqrt{5}) = \boxed{7}.$$

16. D

A prime occurring to exponent a in n allows $a+1$ choices for its exponent in a divisor. These choices are independent for different primes. To obtain six choices in total, the exponents must be 5, or 2 and 1. Hence

$$n = p^5 \qquad \text{or} \qquad n = p^2 q,$$

where p and q are distinct primes in the second form.

For $n = p^5$, the divisor sums at $p = 2$ and $p = 3$ are respectively 63 and 364. Every other prime has $p \geq 5$, giving $n \geq 5^5 > 434$. Thus this form gives no solution.

For $n = p^2 q$, the six divisors give

$$(1 + p + p^2)(1 + q) = 434 = 2 \times 7 \times 31.$$

The factor $1 + p + p^2$ is odd and is at least 7. The odd divisors of 434 are $1, 7, 31, 217$. The value 1 is too small, and 217 would force $q + 1 = 2$, so $q = 1$, which is not prime. Hence there are only two cases:

- $p^2 + p + 1 = 7$ gives $p = 2$ and $q + 1 = 62$, so $q = 61$ and $n = 244$;
- $p^2 + p + 1 = 31$ gives $p = 5$ and $q + 1 = 14$, so $q = 13$ and $n = 325$.

Both pairs consist of distinct primes, so each number has exactly six divisors. Their divisor sums are respectively $7 \times 62 = 434$ and $31 \times 14 = 434$, so both are attained.

The required sum is therefore

$$244 + 325 = \boxed{569}.$$

17. C

Let the "add 3" rule produce the rth number, and the "subtract 1" rule produce the sth number. Each changes the value by 2 from what the usual rule would produce, with opposite signs.

Each later usual update doubles any existing error. The correct tenth term is $2^{10} - 1 = 1023$, so

$$1047 - 1023 = 2^{11-r} - 2^{11-s} = 24.$$

The positive difference forces $r < s$. Factoring gives

$$24 = 2^{11-s}(2^{s-r} - 1).$$

The bracket is odd. Since $24 = 8 \times 3$ and 3 is odd, we must have

$$2^{11-s} = 8, \qquad 2^{s-r} - 1 = 3.$$

Thus $s = 8$ and $s - r = 2$, giving $r = 6$. The preceding term is correct and equals $u_5 = 31$. Therefore the first incorrect number is

$$2 \times 31 + 3 = \boxed{65}.$$

These positions are unique and valid. After 31, the student's list is $65, 131, 261, 523, 1047$, where the second exceptional rule gives $261 = 2(131) - 1$.

18. B

The midpoint of AB is

$$M = \left(\frac{5}{2}, 4\right),$$

and the displacement from A to B is $(3, 4)$.

The centre of a square is half a side length from the midpoint of a side, in a perpendicular direction. A perpendicular displacement of half the length of $(3, 4)$ is

$$\left(-2, \frac{3}{2}\right)$$

or its negative. Thus the two possible centres are

$$\left(\frac{1}{2}, \frac{11}{2}\right) \quad \text{and} \quad \left(\frac{9}{2}, \frac{5}{2}\right).$$

Only the first has x-coordinate less than 1, so the required centre is

$$\boxed{\left(\frac{1}{2}, \frac{11}{2}\right)}.$$

19. F

Write $f(x) = ax^2 + bx + c$. The tangent condition gives

$$f(1) = 3, \qquad f'(1) = 2,$$

so $a + b + c = 3$ and $2a + b = 2$. Therefore $b = 2 - 2a$, $c = 1 + a$, and

$$f(x) = 2x + 1 + a(x - 1)^2.$$

Integrating the expanded polynomial over $0 \le x \le 2$ gives

$$10 = \int_0^2 (2x + 1)\,dx + a \int_0^2 (x^2 - 2x + 1)\,dx = 6 + \frac{2a}{3}.$$

Hence $a = 6$. Since $f'(x) = 2 + 2a(x - 1)$,

$$f'(2) = 2 + 12 = \boxed{14}.$$

20. D

Write

$$p(x) = (x-1)^2 + 2, \qquad q(x) = 10 - (x+1)^2.$$

Their difference is

$$p(x) - q(x) = 2x^2 - 6.$$

Thus $m(x) = p(x)$ when $-\sqrt{3} \le x \le \sqrt{3}$, and $m(x) = q(x)$ outside these bounds.

On $-\sqrt{3} \le x \le \sqrt{3}$, the greatest distance from x to 1 is at $x = -\sqrt{3}$. Therefore the greatest value of $p(x)$ there is

$$p(-\sqrt{3}) = 6 + 2\sqrt{3}.$$

For $x \le -\sqrt{3}$, we have $|x+1| \ge \sqrt{3} - 1$, so

$$q(x) \le 10 - (\sqrt{3} - 1)^2 = 6 + 2\sqrt{3}.$$

For $x \ge \sqrt{3}$, we have $|x+1| \ge \sqrt{3} + 1$, so

$$q(x) \le 10 - (\sqrt{3} + 1)^2 = 6 - 2\sqrt{3}.$$

The global upper bound is attained at $x = -\sqrt{3}$, where $p(x) = q(x)$. Hence the answer is

$$\boxed{6 + 2\sqrt{3}}.$$