1. What is the value of

$$\frac{3^{21} - 3^{20}}{3^{20} + 3^{19}}\,?$$

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{2}$

D 2

E 6

2. Consider the following claim about positive numbers a, b and c:

If a, b and c are the side lengths of a triangle, then a^2, b^2 and c^2 are also the side lengths of a triangle.

Here a triangle must have three non-collinear vertices. Which of the following is a counterexample?

A $a = 1, \quad b = 2, \quad c = 3$

B $a = 2, \quad b = 3, \quad c = 3$

C $a = 3, \quad b = 3, \quad c = 4$

D $a = 2, \quad b = 3, \quad c = 4$

E $a = 1, \quad b = 2, \quad c = 4$

3. Consider the statement:

If a quadrilateral is a square, then its diagonals are perpendicular and equal in length.

Which of the following is the contrapositive of this statement?

A If a quadrilateral has diagonals that are not perpendicular or are not equal in length, then it is not a square.

B If a quadrilateral has diagonals that are perpendicular and equal in length, then it is a square.

C If a quadrilateral is not a square, then its diagonals are not perpendicular or are not equal in length.

D If a quadrilateral has diagonals that are not perpendicular and are not equal in length, then it is not a square.

E If a quadrilateral has diagonals that are not perpendicular or are not equal in length, then it is a square.

4. How many integer values of x satisfy

$$x = 2\sin\left(\frac{\pi x}{6}\right)?$$

A 1

B 5

C 2

D 4

E 3

5. Let b and c be integers. Consider the statements

$$P: \quad b^2 - 8c \text{ is the square of a positive integer,}$$
$$Q: \quad 2x^2 + bx + c = 0 \text{ has two distinct integer solutions.}$$

Which of the following correctly describes P as a condition for Q?

A Both necessary and sufficient

B Necessary but not sufficient

C Sufficient but not necessary

D Neither necessary nor sufficient

6. How many integer values of x satisfy

$$\log_{1/2}(x^2) \geq \log_{1/2}(2x+3)?$$

A 1

B 2

C 3

D 5

E Infinitely many

F 4

7. The curve C has equation

$$y = \frac{x^4 - 9}{x^2 + 3}.$$

The tangent to C at the point where $x = 1$ meets the coordinate axes at A and B.

What is the area of triangle OAB, where O is the origin?

A 1

B 2

C $\frac{9}{2}$

D 4

E 8

8. Let a and b be positive real numbers. Consider the statements

P: $4ab > 1$.

Q: The line $y = ax + b$ and the curve $y = \sqrt{x}$ have no common point.

Which of the following correctly describes P as a condition for Q?

A P is both necessary and sufficient for Q.

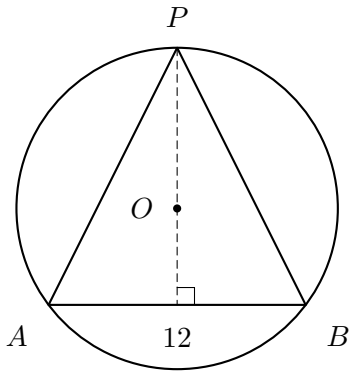
B P is necessary but not sufficient for Q.

C P is sufficient but not necessary for Q.

D P is neither necessary nor sufficient for Q.

9. The points A, B and P lie on a circle with centre O. The chord AB has length 12, and P lies on the major arc AB with $PA = PB$.

The area of triangle PAB is 72. What is the radius of the circle?



A 6

B $\frac{15}{2}$

C 12

D $6\sqrt{5}$

E 15

10. A collection contains at least two distinct points in a plane. Consider the statement:

> Every point in the collection has another point in the collection less than 1 unit away, and another point in the collection more than 2 units away.

Which of the following is the logical negation of this statement? In each option, "other points" means other points in the collection.

A There is a point for which every other point is at least 1 unit away and every other point is at most 2 units away.

B For every point, either every other point is at least 1 unit away or every other point is at most 2 units away.

C There is a point with another point at least 1 unit away and another point at most 2 units away.

D There is a point for which either every other point is at least 1 unit away or every other point is at most 2 units away.

E There is a point for which either every other point is more than 1 unit away or every other point is less than 2 units away.

11. A sequence is defined by

$$u_1 = 2, \qquad u_2 = 1, \qquad u_{n+2} = 3u_n \quad \text{for every integer } n \geq 1.$$

What is the value of $u_1 + u_2 + \cdots + u_{20}$?

A $\dfrac{3^{10} - 1}{2}$

B $3^{10} - 1$

C $\dfrac{3(3^9 - 1)}{2}$

D $3^{20} - 1$

E $\dfrac{3(3^{10} - 1)}{2}$

12. Three distinct positive real numbers form one data set. A second data set is obtained by taking the reciprocal of each number in the first set.

Consider the statements

P: The mean of the first data set is greater than its median.

Q: The mean of the second data set is greater than its median.

Which of the following correctly describes P as a condition for Q?

A P is sufficient but not necessary for Q.

B P is neither necessary nor sufficient for Q.

C P is necessary but not sufficient for Q.

D P is both necessary and sufficient for Q.

13. A student gives the following argument about

$$f(x) = x - \frac{1}{x}, \qquad x \neq 0.$$

I. Differentiating gives $f'(x) = 1 + \dfrac{1}{x^2}$.

II. Hence $f'(x) > 0$ for every nonzero real number x.

III. Therefore, for any nonzero real numbers $a < b$, we have $f(a) < f(b)$.

IV. It follows that the equation $f(x) = 0$ has at most one real solution.

Which of the following best describes the argument?

A The first error is in line I.

B The first error is in line II.

C The first error is in line III.

D The first error is in line IV.

E The argument is completely correct.

14. A bag contains two red balls and three blue balls. Two balls are drawn at random, one after the other without replacement, and their colours are recorded. Both balls are then returned to the bag.

This procedure is repeated independently, giving two more recorded colours. At each draw, every ball then in the bag is equally likely to be chosen.

Given that the four recorded colours consist of two reds and two blues, what is the probability that each pair drawn contained one ball of each colour?

A $\frac{9}{25}$

B $\frac{3}{5}$

C $\frac{2}{3}$

D $\frac{6}{7}$

E $\frac{12}{13}$

15. Let a and b be real constants. Consider the statements

P: $a \geq b^2$.

Q: $x^4 + ax^2 + bx + 1 > 0$ for every real number x.

Which of the following is correct?

A P is both necessary and sufficient for Q

B P is sufficient but not necessary for Q

C P is necessary but not sufficient for Q

D P is neither necessary nor sufficient for Q

16. For which real values of k does the equation

$$\sin\theta + \cos\theta + |\sin\theta - \cos\theta| = k$$

have exactly two distinct solutions satisfying $0 \leq \theta < 2\pi$?

A $-\sqrt{2} < k < \sqrt{2}$

B $-\sqrt{2} \leq k < \sqrt{2}$ or $k = 2$

C $-\sqrt{2} < k \leq \sqrt{2}$ or $k = 2$

D $-\sqrt{2} < k < \sqrt{2}$ or $k = 2$

E $\sqrt{2} < k < 2$

17. The polynomial

$$f(x) = x^3 + ax^2 + bx + c$$

has integer coefficients, and both $f(0)$ and $f(1)$ are odd integers.

You may use the fact that any rational root of a polynomial with integer coefficients and leading coefficient 1 is an integer.

Consider the statements

I. $f(n)$ is odd for every integer n.

II. Every real root of $f(x) = 0$ is irrational.

III. The equation $f(x) = 0$ has exactly one real root.

Which of these statements must be true?

A None of them

B I only

C II only

D III only

E I and III only

F I and II only

G II and III only

H I, II and III

18. A train is 120 m long. At a constant speed it takes 30 seconds to pass completely past a straight platform.

At three-quarters of that speed, it takes 64 seconds to pass completely past a second straight platform, whose length is twice that of the first.

Each time is measured from the instant the front of the train reaches the near end of the platform until the rear leaves the far end.

What is the length of the first platform?

A 90 m

B 300 m

C 360 m

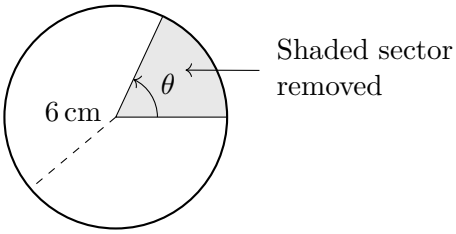
D 480 m

E 180 m

19. A sector of angle θ radians is removed from a circular sheet of radius 6 cm, where $0 < \theta < 2\pi$. The straight edges of the remaining sector are joined, without overlap, to form the curved surface of a right circular cone.

What value of θ gives the greatest volume of the cone?

The volume of a cone of base radius r and perpendicular height h is $V = \frac{1}{3}\pi r^2 h$.



Not to scale

A $2\pi\left(1 - \dfrac{1}{\sqrt{3}}\right)$

B $2\pi\sqrt{\dfrac{2}{3}}$

C $2\pi\left(1 - \sqrt{\dfrac{2}{3}}\right)$

D $\pi\left(1 - \sqrt{\dfrac{2}{3}}\right)$

E $2\pi\left(1 - \dfrac{1}{\sqrt{2}}\right)$

20. Ten distinct real numbers are written in a row. A *subsequence* is a choice of some of these numbers that keeps their original order.

The following proof shows that there is a strictly increasing or strictly decreasing subsequence of at least four terms.

Assume that no such subsequence exists. Write the numbers as $a_1, \ldots, a_{10}$.

[Missing argument]

Thus we have ten different pairs (r_i, s_i), with each coordinate equal to 1, 2 or 3. However, there are only $3 \times 3 = 9$ possible pairs, a contradiction.

Which option correctly completes the proof?

A Let r_i, s_i be the greatest lengths of strictly increasing and strictly decreasing subsequences ending at a_i, respectively. For $i < j$, appending a_j gives $r_j > r_i$ if $a_i < a_j$, and $s_j > s_i$ if $a_i > a_j$.

B Let r_i be one plus the number of earlier terms smaller than a_i, and s_i one plus the number larger. Neither label exceeds 3, and $r_i + s_i = i + 1$ makes the pairs distinct.

C Let r_i, s_i be the greatest lengths of strictly increasing and strictly decreasing blocks of consecutive terms ending at a_i, respectively. For any $i < j$, appending a_j extends one such block, so a label increases.

D Let r_i, s_i be the greatest lengths of strictly increasing subsequences ending and starting at a_i, respectively. For $i < j$, either $r_j > r_i$ or $s_i > s_j$, according to the order of a_i, a_j.

E Let r_i, s_i be the greatest lengths of strictly increasing and strictly decreasing subsequences among $a_1, \ldots, a_i$, respectively. Moving from i to any later j introduces a new term, so at least one label increases.