1. C

Factor out 3^{19} from both numerator and denominator:

$$\frac{3^{21}-3^{20}}{3^{20}+3^{19}} = \frac{3^{19}(9-3)}{3^{19}(3+1)} = \frac{6}{4} = \boxed{\frac{3}{2}}.$$

2. D

A counterexample must satisfy the premise and fail the conclusion. For option D,

$$2 + 3 > 4,$$

so 2, 3 and 4 form a triangle. Their squares do not, since

$$2^2 + 3^2 = 13 < 16 = 4^2.$$

In A, $1 + 2 = 3$, so the original lengths give a degenerate configuration, not a triangle. In E, $1 + 2 < 4$, so the premise also fails. In B and C, the squared lengths do form triangles, since respectively $4 + 9 > 9$ and $9 + 9 > 16$. Thus D is the unique counterexample.

3. A

The contrapositive reverses an implication and negates both parts.

The negation of "the diagonals are perpendicular and equal in length" is "the diagonals are not perpendicular or are not equal in length". The word "or" includes the possibility that both properties fail.

The required conclusion is that the quadrilateral is not a square. This is precisely option A. Option B is the converse; C is the inverse; D incorrectly uses "and" in the negated premise; E fails to negate the conclusion.

4. E

Since a sine value lies between -1 and 1, every solution satisfies $-2 \le x \le 2$.

Both sides are odd functions of x, so a nonzero solution occurs together with its negative. It is therefore enough to check $x = 0, 1, 2$:

x	0	1	2
$2\sin(\pi x/6)$	0	1	$\sqrt{3}$

Thus 0 and 1 work, while 2 does not. By symmetry, -1 works and -2 does not. The integer solutions are -1, 0 and 1, giving $\boxed{3}$ solutions.

5. B

Suppose Q holds, and let r be one of its integer roots. Since

$$2r^2 + br + c = 0,$$

we have

$$b^2 - 8c = b^2 + 16r^2 + 8br = (b + 4r)^2.$$

The discriminant is positive because the two roots are distinct. Thus it is the square of a positive integer, and P is necessary for Q.

It is not sufficient. Take $b = -5$ and $c = 2$. Then

$$b^2 - 8c = 25 - 16 = 9,$$

so P holds, but

$$2x^2 - 5x + 2 = (2x - 1)(x - 2)$$

has roots $\frac{1}{2}$ and 2, which are not both integers. Hence the answer is B.

6. F

The logarithms require $x \neq 0$ and $2x + 3 > 0$.

Because $0 < \frac{1}{2} < 1$, the logarithm to this base is decreasing. Therefore, within its domain, the inequality is equivalent to

$$x^2 \leq 2x + 3, \qquad (x + 1)(x - 3) \leq 0,$$

which gives $-1 \leq x \leq 3$.

Throughout these inequalities, $2x + 3 \geq 1 > 0$, so the only remaining domain exclusion is $x = 0$. The allowed integers are

$$-1, \quad 1, \quad 2, \quad 3.$$

There are $\boxed{4}$. The endpoints are included because the original inequality is non-strict.

7. D

Since $x^2 + 3 > 0$ for every real x, we can simplify the equation without losing any points:

$$y = \frac{(x^2 - 3)(x^2 + 3)}{x^2 + 3} = x^2 - 3.$$

At $x = 1$, $y = -2$ and the tangent gradient is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x = 2.$$

Thus the tangent is

$$y + 2 = 2(x - 1), \qquad y = 2x - 4.$$

Its intercepts are $(2, 0)$ and $(0, -4)$. The lengths of the perpendicular sides of triangle OAB are 2 and 4, so its area is

$$\frac{1}{2} \times 2 \times 4 = \boxed{4}.$$

8. A

At a common point, put $t = \sqrt{x}$, so $t \geq 0$ and $x = t^2$. The intersection equation becomes

$$at^2 - t + b = 0.$$

Its discriminant is $1 - 4ab$. If $4ab > 1$, this equation has no real roots, so the two graphs have no common point. Thus P is sufficient for Q.

If $4ab \leq 1$, the quadratic has real roots. Their product is $b/a > 0$ and their sum is $1/a > 0$, so both roots are positive, including the repeated-root case. Each therefore gives a valid intersection with $x = t^2$. Hence Q fails whenever P fails, so P is also necessary.

The strict boundary matters: when $4ab = 1$, there is one common point. The answer is A.

9. B

Let M be the midpoint of AB. Since $PA = PB$, the line PM is perpendicular to AB. The centre O is also on this perpendicular bisector.

The area gives

$$\frac{1}{2} \times 12 \times PM = 72, \qquad PM = 12.$$

Write the radius as r. Since P is on the major arc and the perpendicular bisector, $PO = r$ and $OM = 12 - r$. Also $AM = 6$. Pythagoras in triangle OMA gives

$$r^2 = (12 - r)^2 + 6^2, \qquad 24r = 180, \qquad r = \boxed{\frac{15}{2}}.$$

This has $OM = 9/2 > 0$ and $AM < r$, consistent with the stated configuration.

10. D

For the original statement to fail, there must be at least one point P for which at least one of the two required witnesses does not exist.

Failure to have another point less than 1 unit away means that every other point is at least 1 unit away. Failure to have another point more than 2 units away means that every other point is at most 2 units away.

Either failure is enough, and both may occur. Thus the required negation is D. The same point P is used when testing its two requirements; the original nearby and distant witnesses need not be the same point.

11. E

Group consecutive terms into pairs. Their sums are

$$u_1 + u_2 = 3, \qquad u_3 + u_4 = 9, \qquad u_5 + u_6 = 27, \quad \ldots.$$

Indeed,

$$u_{2k+1} + u_{2k+2} = 3(u_{2k-1} + u_{2k}).$$

There are 10 pairs, so their sums form a geometric progression with first term 3 and common ratio 3. Therefore

$$u_1 + \cdots + u_{20} = 3\frac{3^{10} - 1}{3 - 1} = \boxed{\frac{3(3^{10} - 1)}{2}}.$$

12. B

For positive numbers $x < y < z$, their reciprocals have order

$$\frac{1}{z} < \frac{1}{y} < \frac{1}{x}.$$

Thus the medians of the two data sets are y and $1/y$.

Take the first data set to be

$$\frac{2}{3}, \quad 1, \quad 3.$$

Its mean is $14/9 > 1$, whereas the reciprocal data set has mean $17/18 < 1$. Both medians are 1, so P holds and Q fails. Hence P is not sufficient for Q.

Now take the first data set to be

$$\frac{1}{3}, \quad 1, \quad \frac{3}{2}.$$

This exchanges the previous two data sets. Its mean is $17/18 < 1$ and the reciprocal mean is $14/9 > 1$, again with both medians 1. Thus Q holds and P fails, so P is not necessary for Q.

The answer is B.

13. C

Lines I and II are valid: write $f(x) = x - x^{-1}$ and apply the power rule.

The positive derivative establishes strict increase separately for $x < 0$ and for $x > 0$. It does not justify comparing inputs on opposite sides of the excluded point 0.

In fact,

$$-1 < 1, \qquad f(-1) = 0 = f(1),$$

so the assertion in line III is false. This is the first error, giving C. Line IV would follow from the strict-increase assertion in line III, but that assertion has not been established and is false.

14. D

For one pair, let M mean that its colours are mixed. The probabilities of the three pair types are

$$\Pr(RR) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}, \qquad \Pr(BB) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10},$$

$$\Pr(M) = 2 \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5}.$$

Both balls are returned before the second pair is drawn, so the two pair types are independent.

There are two ways to record two reds and two blues in total:

- Both pairs are mixed, with probability $(3/5)^2 = 9/25$.
- One pair is RR and the other is BB, in either order, with probability $2(1/10)(3/10) = 3/50$.

These cases are disjoint and exhaust the given event. The required conditional probability is therefore

$$\frac{9/25}{9/25 + 3/50} = \frac{18}{21} = \boxed{\frac{6}{7}}.$$

15. B

Suppose that P holds. Completing a square gives

$$x^4 + ax^2 + bx + 1 = x^4 + (a - b^2)x^2 + \left(bx + \frac{1}{2}\right)^2 + \frac{3}{4}$$
$$\geq \frac{3}{4} > 0.$$

This is valid for every real x, since $a - b^2 \geq 0$. Thus P is sufficient for Q, including when $a = b^2$.

To test necessity, take $a = -1$ and $b = 0$. Then

$$x^4 - x^2 + 1 = \left(x^2 - \frac{1}{2}\right)^2 + \frac{3}{4} > 0$$

for every real x, so Q holds. However, $a = -1 < 0 = b^2$, so P fails.

Consequently P is sufficient but not necessary for Q.

16. D

For any real u, v, the expression $u+v+|u-v|$ is twice the larger of u, v. Put $t = k/2$. We therefore require the larger of $\sin\theta$ and $\cos\theta$ to equal t.

On the unit circle, the possible points are

$$(\cos\theta, \sin\theta) = (t,\, \pm\sqrt{1-t^2}) \quad \text{or} \quad (\pm\sqrt{1-t^2},\, t),$$

retaining only points whose other coordinate is at most t. Count distinct points, since each corresponds to one angle in the stated domain.

- If $-1/\sqrt{2} < t < 1/\sqrt{2}$, only the negative square-root coordinate is allowed. It gives two distinct points. At $t = -1/\sqrt{2}$ these merge, leaving one; at $t = 1/\sqrt{2}$ the shared positive-coordinate point also appears, giving three.
- If $1/\sqrt{2} < t < 1$, both signs are allowed, giving four distinct points.
- At $t = 1$, the square-root signs coincide, leaving exactly $(1, 0)$ and $(0, 1)$. Values outside $-1/\sqrt{2} \le t \le 1$ give no points.

Thus exactly two solutions occur for

$$\boxed{-\sqrt{2} < k < \sqrt{2} \quad \text{or} \quad k = 2}.$$

The half-open angular domain counts $(1, 0)$ once, at $\theta = 0$.

17. F

If n is even, every term of $f(n)$ except c is even. Since $c = f(0)$ is odd, $f(n)$ is odd.

If n is odd, each power of n is odd. Hence $f(n)$ has the same parity as

$$1 + a + b + c = f(1),$$

which is odd. This also applies to negative integers, so I is true.

It follows that no integer is a root, because 0 is even. By the supplied fact, a rational root would have to be an integer. Thus every real root is irrational, proving II.

For III, consider

$$f(x) = x^3 - 3x + 1.$$

It satisfies $f(0) = 1$ and $f(1) = -1$, both odd. But

$$f(-2) = -1, \quad f(-1) = 3, \quad f(1) = -1, \quad f(2) = 3.$$

Its graph crosses the axis between each consecutive pair of these inputs, giving three distinct real roots. A cubic has at most three, so III is false.

Only I and II must be true.

18. E

Let the first platform have length L metres, and let the first speed be v metres per second. During each measured time, the train travels the platform length plus its own length. Therefore

$$30v = L + 120, \qquad 64\left(\frac{3}{4}v\right) = 2L + 120.$$

Twice the first equation minus the second gives

$$60v - 48v = 240 - 120,$$

so $v = 10$. Substitution in the first equation gives

$$L = 30(10) - 120 = \boxed{180\,\text{m}}.$$

As a check, the second platform is 360 m long. The train travels 480 m at $7.5\,\text{m}\,\text{s}^{-1}$, taking 64 seconds.

19. C

The sheet's radius becomes the cone's slant height, so

$$r^2 + h^2 = 36.$$

Using the height as the variable avoids differentiating a square root:

$$V = \frac{\pi}{3}(36 - h^2)h = \frac{\pi}{3}(36h - h^3), \qquad 0 < h < 6.$$

Thus

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi(12 - h^2).$$

The volume increases for $0 < h < 2\sqrt{3}$ and decreases for $2\sqrt{3} < h < 6$. Its greatest value is attained when $h = 2\sqrt{3}$, giving

$$r^2 = 36 - 12 = 24, \qquad r = 2\sqrt{6}.$$

The retained arc becomes the base circumference, so

$$6(2\pi - \theta) = 2\pi r = 4\pi\sqrt{6}.$$

Therefore the angle removed is

$$\boxed{\theta = 2\pi\left(1 - \sqrt{\frac{2}{3}}\right)}.$$

This satisfies $0 < \theta < 2\pi$, so the maximizing cone is admissible.

20. A

Use the two greatest ending lengths in A. Each exists because the row is finite. A single term is a subsequence, and the contrary assumption excludes lengths of four or more, so

$$1 \le r_i \le 3, \qquad 1 \le s_i \le 3.$$

For $i < j$, there are exactly two cases:

- If $a_i < a_j$, append a_j to a longest increasing subsequence ending at a_i. The indices remain in order, so $r_j \ge r_i + 1 > r_i$.
- If $a_i > a_j$, do the same with a longest decreasing subsequence, giving $s_j \ge s_i + 1 > s_i$.

Thus at least one coordinate differs for every pair of positions. Ten different pairs cannot fit into the nine possibilities, completing the contradiction.

The alternative constructions fail locally:

- B: in the row $3, 2, 1, 4$, the fourth label r_4 is 4, although there is no four-term monotone subsequence. Counts of earlier terms are not lengths of ordered subsequences.
- C: in $2, 4, 1, 3$, the second and fourth pairs are both $(2, 1)$. Appending across a gap need not preserve consecutiveness.
- D: in $2, 1$, both increasing ending and starting lengths are 1 at each position, so the two pairs agree.
- E: in $2, 4, 1, 3$, the third and fourth prefix pairs are both $(2, 2)$. A new term need not increase either maximum.

These short rows test the claimed local inferences; none contains a four-term monotone subsequence.