1. C

Every side of the rhombus has length $20/4 = 5$ cm. In the right-angled triangle PBQ, we have $PB = 3$ cm, so

$$BQ^2 = PQ^2 - PB^2 = 25 - 9 = 16.$$

Thus $BQ = 4$ cm and, since Q is a midpoint, $BC = 8$ cm. The area of the rectangle is therefore

$$6 \times 8 = \boxed{48 \text{ cm}^2}.$$

2. B

Since

$$f'(x) = 3(x-1)(x-5),$$

the stationary points have x-coordinates 1 and 5. Integrating gives

$$f(x) = x^3 - 9x^2 + 15x + C.$$

Hence $f(1) = 7 + C$ and $f(5) = -25 + C$. The arbitrary constant cancels when finding the gradient:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{-32}{4} = \boxed{-8}.$$

The derivative changes sign at both roots, so these are indeed the two stationary points.

3. B

The January charge for water usage is $30-6=24$ pounds. February usage is 75% of January usage, so its usage charge is

$$\frac{3}{4}\times 24=18$$

pounds. Adding the unchanged fixed charge gives a February bill of

$$18+6=\boxed{24}$$

pounds.

4. D

Write the line as $y = mx - 3$, and let the intersection x-coordinates be p and q. They are the roots of

$$x^2 - mx + 3 = 0.$$

Comparing this with $(x - p)(x - q)$ gives $pq = 3$. The required sum of y-coordinates is therefore

$$p^2 + q^2 = (p + q)^2 - 2pq = 4^2 - 2 \times 3 = \boxed{10}.$$

The given sum also gives $m = 4$, and the intersection x-coordinates are 1 and 3, confirming that two distinct real intersections exist.

5. C

The two given differences are

$$15a + 3b = 15, \qquad 33a + 3b = 33.$$

Subtracting gives $18a = 18$, so $a = 1$. Substituting back gives $b = 0$. Therefore

$$u_{10} - u_7 = (100 - 49)a + 3b = \boxed{51}.$$

Equivalently, the increases over successive blocks of three indices have constant difference 18, so the next increase is $33 + 18 = 51$. The value of c is irrelevant.

6. E

Let $AB = u$ and $AC = v$. The area formula gives

$$\frac{1}{2}uv\sin 60^\circ = 6\sqrt{3}, \qquad \text{so} \qquad uv = 24.$$

By the cosine rule,

$$49 = u^2 + v^2 - 2uv\cos 60^\circ = u^2 + v^2 - uv,$$

so $u^2 + v^2 = 73$. Hence

$$(u+v)^2 = 73 + 48 = 121.$$

Lengths are positive, so $u + v = 11$ and the perimeter is $11 + 7 = \boxed{18}$.

For existence, the two adjacent sides are 3 and 8. These, together with 7, satisfy the strict triangle inequalities and have the stated area and included angle.

7. C

Put $t = \sqrt{x}$ and $u = \sqrt{5 - x}$. Then

$$t \geq 0, \qquad u \geq 0, \qquad t^2 + u^2 = 5, \qquad t + 2u = k.$$

Every such pair gives exactly one $x = t^2$. Necessarily $k > 0$. Eliminating $u = (k - t)/2$ gives

$$5t^2 - 2kt + k^2 - 20 = 0, \qquad t = \frac{k \pm 2\sqrt{25 - k^2}}{5}.$$

Real candidates require $k \leq 5$. A candidate is valid precisely when $0 \leq t \leq k$, which makes both t and u nonnegative.

For the plus sign, $t > 0$, and $t \leq k$ is equivalent to $\sqrt{25 - k^2} \leq 2k$, or $k \geq \sqrt{5}$. Thus this branch is valid for $\sqrt{5} \leq k \leq 5$.

For the minus sign, $t \leq k$ automatically, and $t \geq 0$ is equivalent to $k \geq 2\sqrt{25 - k^2}$, or $k \geq 2\sqrt{5}$. Thus this branch is valid for $2\sqrt{5} \leq k \leq 5$.

The candidates are distinct when $k < 5$, and coincide when $k = 5$. Therefore exactly one solution occurs when

$$\boxed{\sqrt{5} \leq k < 2\sqrt{5} \quad \text{or} \quad k = 5}.$$

At the lower endpoint $k = \sqrt{5}$, the solution is $x = 5$; at $k = 2\sqrt{5}$, there are two solutions, $x = 0$ and $x = 16/5$; at $k = 5$, the unique solution is $x = 1$.

8. C

Write a contest history using the winner of each game. The histories in which A wins the contest are AA, ABA and BAA; play stops as soon as either player has two wins.

Their probabilities are

$$P(AA) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}, \qquad P(ABA) = P(BAA) = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{27}.$$

The last two histories each require three games. Therefore the required conditional probability is

$$\frac{4/27 + 4/27}{4/9 + 4/27 + 4/27} = \frac{8}{20} = \boxed{\frac{2}{5}}.$$

9. D

Integrating the second derivative gives

$$f'(x) = 3x^2 + C.$$

Since $f'(1) = 2$, we have $3 + C = 2$, so $C = -1$.

A horizontal tangent has gradient zero. Hence

$$3x^2 - 1 = 0, \qquad x = -\frac{1}{\sqrt{3}} \quad \text{or} \quad x = \frac{1}{\sqrt{3}}.$$

Both values are in the domain. Their positive difference is

$$\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right) = \boxed{\frac{2}{\sqrt{3}}}.$$

10. E

The arithmetic-progression condition is

$$\binom{n}{4} + \binom{n}{6} = 2\binom{n}{5}.$$

Since $n \geq 6$, the coefficient $\binom{n}{4}$ is positive, so division by it is valid. The successive coefficient ratios give

$$1 + \frac{(n-4)(n-5)}{30} = \frac{2(n-4)}{5}.$$

Multiplying by 30 and rearranging,

$$30 + (n-4)(n-5) = 12(n-4), \qquad n^2 - 21n + 98 = (n-7)(n-14) = 0.$$

Both roots satisfy $n \geq 6$. For $n = 7$, the coefficients are $35, 21, 7$, a decreasing arithmetic progression. For $n = 14$, they are $1001, 2002, 3003$, an increasing arithmetic progression. Thus both are valid and the required sum is

$$\boxed{7 + 14 = 21}.$$

11. C

The divisibility and leading coefficient allow us to write

$$P(x) = (x^2 + 1)(x^2 + ax + b)$$

for real constants a and b. Expanding gives

$$P(x) = x^4 + ax^3 + (b + 1)x^2 + ax + b.$$

On division by $x^2 - 1$, the remainders of x^2, x^3 and x^4 are respectively 1, x and 1. The remainder of $P(x)$ is therefore

$$2ax + 2b + 2.$$

Comparing with $4x + 6$ gives $a = 2$ and $b = 2$. Consequently

$$P(2) = (4 + 1)(4 + 4 + 2) = \boxed{50}.$$

12. D

The centre of either circle lies on the perpendicular bisector of the segment joining $(1,0)$ and $(5,0)$, so write it as $(3,k)$. Its squared radius is

$$r^2 = (3-1)^2 + k^2 = 4 + k^2.$$

The perpendicular from $(3,k)$ to $y = x$ has equation $y - k = -(x-3)$, so its foot is

$$T = \left(\frac{3+k}{2}, \frac{3+k}{2}\right).$$

Tangency means that the squared distance from the centre to T equals r^2. Thus

$$\frac{(3-k)^2}{2} = 4 + k^2, \qquad k^2 + 6k - 1 = 0, \qquad k = -3 \pm \sqrt{10}.$$

These are distinct real centres, and both give positive radii and tangency to the specified line. Their squared radii are

$$4 + (-3+\sqrt{10})^2 = 23 - 6\sqrt{10}, \qquad 4 + (-3-\sqrt{10})^2 = 23 + 6\sqrt{10}.$$

The required sum of areas is therefore

$$\pi\big((23 - 6\sqrt{10}) + (23 + 6\sqrt{10})\big) = \boxed{46\pi}.$$

13. C

Multiplying each class width by its frequency density gives frequencies

$$4 \cdot 2 = 8, \qquad 8 \cdot 1 = 8, \qquad 12 \cdot 1 = 12.$$

The total frequency is 28, so the quartiles correspond to cumulative frequencies 7 and 21.

The lower quartile lies in the first class. Linear interpolation gives

$$Q_1 = 0 + \frac{7}{8} \cdot 4 = \frac{7}{2}.$$

The cumulative frequency below 12 is 16, so the upper quartile lies in the final class and is

$$Q_3 = 12 + \frac{21 - 16}{12} \cdot 12 = 17.$$

Hence the estimated interquartile range is

$$Q_3 - Q_1 = 17 - \frac{7}{2} = \boxed{\frac{27}{2} \text{ minutes}}.$$

14. D

Since $a > 0$, the maximum and minimum give

$$c + a = 7, \qquad c - a = 1.$$

Thus $c = 4$ and $a = 3$. Since $b > 0$, the least positive period of $\cos(bx)$ is $2\pi/b$, so

$$\frac{2\pi}{b} = \frac{2\pi}{3}, \qquad b = 3.$$

Therefore

$$f\left(\frac{\pi}{9}\right) = 3\cos\left(\frac{\pi}{3}\right) + 4 = \frac{3}{2} + 4 = \boxed{\frac{11}{2}}.$$

15. D

Write $f(x) = x^4 - 6x^2$. Since $f'(a) = 4a^3 - 12a$, the tangent is

$$y = (4a^3 - 12a)x - 3a^4 + 6a^2.$$

Subtracting its height from the curve's height and factoring gives

$$\begin{aligned} f(x) - ((4a^3 - 12a)x - 3a^4 + 6a^2) &= (x - a)^2(x^2 + 2ax + 3a^2 - 6) \\ &= (x - a)^2((x + a)^2 + 2a^2 - 6). \end{aligned}$$

If $a^2 \geq 3$, both factors are nonnegative for every real x, so the condition holds.

If $a^2 < 3$ and $a \neq 0$, take $x = -a$. The first factor is positive and the second is negative, so the condition fails. If $a = 0$, taking $x = 1$ gives a height difference of -5, so this case also fails.

When $a = \pm\sqrt{3}$, the difference is $(x - a)^2(x + a)^2$, so equality at a second point is allowed. The complete answer is

$$\boxed{a \leq -\sqrt{3} \quad \text{or} \quad a \geq \sqrt{3}}.$$

16. B

Choosing r copies of $-1/x$ gives the term

$$\binom{6}{r}(2x^2)^{6-r}\left(-\frac{1}{x}\right)^r = \binom{6}{r}2^{6-r}(-1)^r x^{12-3r}.$$

Its power of x is negative precisely when $12-3r<0$, so only $r=5$ and $r=6$ contribute.

The two coefficients are

$$\binom{6}{5}2(-1)^5 = -12, \qquad \binom{6}{6}(-1)^6 = 1.$$

Their sum is $\boxed{-11}$. The term with $r=4$ is constant and is not included.

17. D

Complete squares to rewrite the left-hand side as

$$(x^2 - 3x + 2)^2 + (kx - 1)^2.$$

For real x and k, this sum is zero if and only if both squares are zero. Thus

$$(x - 1)(x - 2) = 0, \qquad kx = 1.$$

If $x = 1$, then $k = 1$. If $x = 2$, then $k = 1/2$. Conversely, each of these pairs makes both squares zero and therefore satisfies the original equation.

These are the only possible values of k, and their sum is

$$\boxed{1 + \frac{1}{2} = \frac{3}{2}}.$$

18. C

Let the hundreds, tens and units digits be a, b and c, respectively. Then

$$a + b + c = 12, \qquad a + c = 2b.$$

Hence $3b = 12$, so $b = 4$ and $a + c = 8$.

The difference between the original number and its reversal is

$$(100a + 10b + c) - (100c + 10b + a) = 99(a - c) = 198.$$

Thus $a - c = 2$. Adding this to $a + c = 8$ gives $2a = 10$, so the hundreds digit is $\boxed{5}$.

The resulting number is 543, whose reversal is 345; their difference is indeed 198.

19. C

Choose coordinates

$$A = (0, 0), \quad B = (6, 0), \quad D = (t, h), \quad C = (t + 2, h), \qquad h > 0.$$

The leg AD has length 4, so

$$t^2 + h^2 = 16.$$

The diagonals have gradients $h/(t + 2)$ and $h/(t - 6)$. Neither can be vertical: the other diagonal would then have to be horizontal, which is impossible since $h > 0$.

Their gradients have product -1, giving

$$h^2 = -(t + 2)(t - 6) = -t^2 + 4t + 12.$$

Combining the two equations cancels t^2:

$$16 = 4t + 12, \qquad t = 1, \qquad h^2 = 15.$$

Thus the height is $\sqrt{15}$, and the area is

$$\boxed{\frac{1}{2}(6 + 2)\sqrt{15} = 4\sqrt{15}}.$$

This configuration exists: taking $D = (1, \sqrt{15})$ and $C = (3, \sqrt{15})$ gives a convex trapezium with the required leg length and perpendicular diagonals. Reflection gives the same area.

20. C

The inner function requires $x \neq -2$. The outer function also requires $f(x) \neq -2$. Solving

$$\frac{2x+3}{x+2} = -2$$

gives $x = -7/4$, which must also be excluded.

On this domain,

$$f(f(x)) = \frac{2(2x+3)+3(x+2)}{(2x+3)+2(x+2)} = \frac{7x+12}{4x+7}.$$

Therefore

$$f(f(x)) > 1 \quad \Longleftrightarrow \quad \frac{3x+5}{4x+7} > 0.$$

The numerator and denominator have the same sign when $x < -7/4$ or $x > -5/3$. The value $x = -2$ still has to be removed from the first part, even though the simplified fraction is defined there. Hence

$$\boxed{x < -2 \quad \text{or} \quad -2 < x < -\frac{7}{4} \quad \text{or} \quad x > -\frac{5}{3}}.$$

The value $x = -5/3$ gives equality rather than the strict inequality, so it is excluded.