1. B

Since a and b are positive, $ab \neq 0$ and the denominator is positive. Multiplying the numerator and denominator by ab gives

$$\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}} = \frac{b-a}{b+a} = \frac{2-3}{2+3} = -\frac{1}{5}.$$

2. E

The converse is formed by exchanging the hypothesis and the conclusion, without negating either.

The original hypothesis is that the sequence is both an arithmetic progression and a geometric progression. Its conclusion is that all the terms are equal. The converse is therefore:

If all the terms are equal, then the sequence is both an arithmetic progression and a geometric progression.

This is option E. Option C is the inverse of the original statement, and option D is its contrapositive.

3. D

If $\cos\theta = 0$, the given equation would also give $\sin\theta = 0$, which is impossible. Dividing by $\cos\theta$ therefore gives

$$\tan\theta = -\frac{1}{\sqrt{3}}.$$

The reference angle is 30°. In the specified range $0^\circ < \theta < 180^\circ$, tangent is negative only in the second quadrant. Hence

$$\theta = 180^\circ - 30^\circ = 150^\circ.$$

This satisfies the original equation because $\cos 150^\circ = -\sqrt{3}/2$ and $\sin 150^\circ = 1/2$.

4. C

A counterexample must satisfy the premise and fail the conclusion.

The positive divisors of 3 are 1 and 3. There are two of them, so the premise is true; however, 3 is odd, so the conclusion is false. Thus $n = 3$ is a counterexample.

The numbers 1, 4 and 9 have respectively 1, 3 and 3 positive divisors, so none satisfies the premise. The number 2 has two positive divisors, but is even, so it satisfies the conclusion as well.

5. C

Both O and P are equidistant from A and B, so OP is the perpendicular bisector of AB and passes through M. Thus $AM = 6$, and Pythagoras' theorem in triangle PMA gives

$$PM = \sqrt{10^2 - 6^2} = 8.$$

The radius OA is perpendicular to the tangent AP. Consequently the right triangles OMA and AMP are similar, with hypotenuses OA and AP and corresponding sides AM and PM.

Writing the radius as r, we obtain

$$\frac{r}{10} = \frac{6}{8}, \qquad r = \frac{15}{2}.$$

6. B

Independence gives probabilities pq for two heads and $(1-p)(1-q)$ for two tails. The probability of agreement is their sum, which is strictly positive under the stated assumptions.

If P holds, then $q = 1-p$, so the probabilities of two heads and two tails are equal. Given agreement, each outcome therefore has probability $1/2$. Thus P is sufficient for Q.

Conversely, if Q holds, then

$$\frac{pq}{pq + (1-p)(1-q)} = \frac{1}{2}.$$

Since the denominator is positive, this gives

$$pq = (1-p)(1-q) = 1 - p - q + pq,$$

so $p + q = 1$. Thus P is also necessary for Q, and the answer is both necessary and sufficient.

7. C

Both logarithms require $x > 0$, and the square root requires $\log_2(8x) \geq 0$. Moreover, the right-hand side is nonnegative, so every solution must satisfy

$$\log_2(x/8) \geq 0, \qquad x \geq 8.$$

Put $t = \log_2 x$. Thus $t \geq 3$, and the logarithm laws give

$$t - 3 = \sqrt{t + 3}.$$

On $t \geq 3$, both sides are nonnegative, so squaring is reversible:

$$(t-3)^2 = t + 3, \qquad t^2 - 7t + 6 = (t-1)(t-6) = 0.$$

The candidate $t = 1$ fails $t \geq 3$, leaving $t = 6$ and hence $x = 2^6 = 64$.

At $x = 64$, the two sides of the original equation are $\log_2 8 = 3$ and $\sqrt{\log_2 512} = \sqrt{9} = 3$. At the discarded candidate $x = 2$, they are -2 and 2, so it is not a solution.

8. C

Lines I and II are valid. In particular,

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b).$$

The first error is in line III: two nonzero numbers can have sum zero. Thus division by $3s$ in line IV is not justified for all allowed pairs.

The claim itself can fail. For example, $a = \sqrt[4]{3}$ and $b = -\sqrt[4]{3}$ are nonzero, while $a + b = a^3 + b^3 = 0$ are rational and $ab = -\sqrt{3}$ is irrational. The answer is C.

9. D

Let $S = u_1 + u_2 + \cdots + u_{10}$. Adding the recurrence equations for $n = 1, 2, \ldots, 10$ gives

$$u_2 + u_3 + \cdots + u_{11} = 2S + (1 + 2 + \cdots + 10).$$

The left-hand side is $S - u_1 + u_{11} = S + 3059$, and $1 + 2 + \cdots + 10 = 55$. Therefore

$$S + 3059 = 2S + 55, \qquad S = \boxed{3004}.$$

The supplied terminal value is consistent with the recurrence: its first eleven terms are

$$1, 3, 8, 19, 42, 89, 184, 375, 758, 1525, 3060.$$

10. B

If P holds, then $x - 1 \geq 1$ and $y - 1 \geq 1$. Hence

$$(x-1)(y-1) \geq 1, \qquad xy - x - y + 1 \geq 1,$$

so $xy \geq x + y$. Thus P is sufficient for Q.

It is not necessary. Take $x = 3$ and $y = 3/2$. Then $xy = x + y = 9/2$, so Q holds, but $y < 2$, so P fails. The answer is B.

11. E

For $x > 0$ the curve has equation $y = 1 - x^{-2}$. It is below the axis when $0 < x < 1$ and above it when $x > 1$. An antiderivative is

$$F(x) = x + \frac{1}{x}.$$

Since $F(a) = F(1/a) = a + 1/a$ and $F(1) = 2$, the total area is

$$-\int_a^1 (1 - x^{-2})\,\mathrm{d}x + \int_1^{1/a} (1 - x^{-2})\,\mathrm{d}x = 2\left(a + \frac{1}{a} - 2\right).$$

Equating this to 4 gives $a + 1/a = 4$, and hence

$$a^2 - 4a + 1 = 0, \qquad a = 2 \pm \sqrt{3}.$$

Only $2 - \sqrt{3}$ satisfies $0 < a < 1$. Thus $a = \boxed{2 - \sqrt{3}}$.

12. A

Suppose Q holds. Choose three of the points, with coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). The circle equations are

$$(x_i - h)^2 + (y_i - k)^2 = r^2 \qquad (i = 1, 2, 3).$$

Subtracting the first equation from the other two gives

$$2(x_i - x_1)h + 2(y_i - y_1)k = x_i^2 + y_i^2 - x_1^2 - y_1^2 \qquad (i = 2, 3).$$

These are two linear equations with integer coefficients. Three distinct points on a circle cannot be collinear, so the equations are independent and have a unique solution. Elimination therefore expresses h and k as ratios of integers. Both are rational, proving that P is necessary for Q.

For the converse, the circle $x^2 + y^2 = 1/4$ has rational centre $(0, 0)$, but it contains no point with integer coordinates: $x^2 + y^2$ would be an integer. Thus P is not sufficient. The answer is A.

13. F

Put $c = \cos\theta$ and $s = \sin\theta$. Using $s^2 + c^2 = 1$, the first inequality becomes

$$2(1 - c^2) < 1 + c, \qquad (2c - 1)(c + 1) > 0.$$

Since $-1 \le c \le 1$, this is equivalent to $c > 1/2$.

The second inequality similarly becomes

$$2(1 - s^2) > 1 + s, \qquad (1 - 2s)(s + 1) > 0,$$

which is equivalent to $-1 < s < 1/2$.

The condition $c > 1/2$ gives $0 \le \theta < \pi/3$ or $5\pi/3 < \theta < 2\pi$. In the first part, $s < 1/2$ restricts the angles to $0 \le \theta < \pi/6$. Every angle in the second part satisfies $-1 < s < 1/2$.

The required length is therefore

$$\frac{\pi}{6} + \left(2\pi - \frac{5\pi}{3}\right) = \boxed{\frac{\pi}{2}}.$$

14. D

The statement fails exactly when at least one function in C has no ordered pair $x < y$ for which both required sign conditions hold.

For that fixed function, every ordered pair $x < y$ must satisfy the negation of

$$f(x) \geq 0 \quad \text{and} \quad f(y) < 0.$$

This negation is

$$f(x) < 0 \quad \text{or} \quad f(y) \geq 0.$$

The function is fixed before the ordered pair is considered; which of the two inequalities holds may depend on the pair. The answer is D.

15. D

The lamps overlap precisely when

$$x < y + 3 \qquad \text{and} \qquad y < x + 2,$$

apart from boundary lines, which have zero area. Thus the overlap region in the 6 by 6 square satisfies

$$-2 < x - y < 3.$$

The two excluded corner triangles have perpendicular side lengths 4 and 3, so the overlap region has area

$$36 - \frac{4^2}{2} - \frac{3^2}{2} = \frac{47}{2}.$$

Lamp X switches off first when $x + 2 < y + 3$, or $x - y < 1$. Together with overlap, the favourable region is

$$-2 < x - y < 1.$$

Its two excluded corner triangles have side lengths 4 and 5, giving area

$$36 - \frac{4^2}{2} - \frac{5^2}{2} = \frac{31}{2}.$$

The required conditional probability is therefore

$$\boxed{\frac{31/2}{47/2} = \frac{31}{47}}.$$

The lamps may switch off after minute 6; the restriction applies only to their switching-on times.

16. B

The side lengths $3, 4, 5$ form an arithmetic progression and a non-degenerate right-angled triangle. Its largest angle is $90°$, not $120°$. Thus P can hold while Q fails, so P is not sufficient for Q.

Conversely, the side lengths $1, 1, \sqrt{3}$ form a non-degenerate triangle because $\sqrt{3} < 2$. If C is the angle opposite its longest side, the cosine rule gives

$$\cos C = \frac{1^2 + 1^2 - (\sqrt{3})^2}{2 \cdot 1 \cdot 1} = -\frac{1}{2}.$$

Thus $C = 120°$. The side lengths are not in arithmetic progression: their two consecutive differences are 0 and $\sqrt{3} - 1$. Hence Q can hold while P fails, so P is not necessary either.

Therefore P is neither necessary nor sufficient for Q.

17. E

At each point of contact, the radius is perpendicular to the straight belt section. Translate the smaller contact radius to start at O. The difference of the two radii is $4-1=3$, so a right triangle has hypotenuse $OP=6$ and one shorter side 3.

Consequently each straight belt section has length

$$\sqrt{6^2-3^2}=3\sqrt{3}.$$

If α is the acute angle between the line of centres and a straight belt section, then

$$\sin\alpha=\frac{3}{6}=\frac{1}{2}, \qquad \alpha=30^\circ.$$

The belt covers an angle $180^\circ+2\alpha=240^\circ$ on the larger wheel and $180^\circ-2\alpha=120^\circ$ on the smaller wheel. Their arc lengths are respectively

$$4\cdot\frac{4\pi}{3}=\frac{16\pi}{3}, \qquad 1\cdot\frac{2\pi}{3}=\frac{2\pi}{3}.$$

Adding both arcs and both straight sections gives

$$\boxed{6\pi+6\sqrt{3}}.$$

The wheels are disjoint because $6>4+1$, so this non-crossing exterior belt configuration exists.

18. C

After the numbers are arranged in increasing order, the median is the mean of the 10th and 11th entries. The only pair of permitted values with mean 4 is 2 and 6. Therefore the 10th entry is 2 and the 11th is 6.

Exactly ten entries are consequently at most 2. Five are 2, so five are 0. The number of entries equal to 6 is then

$$20 - 5 - 5 - 3 = 7.$$

The mean is

$$\frac{5 \cdot 0 + 5 \cdot 2 + 7 \cdot 6 + 3 \cdot 9}{20} = \boxed{\frac{79}{20}}.$$

These frequencies place the five 2s in positions 6 to 10 and the seven 6s in positions 11 to 17, so they do give the required median.

19. D

The first list has $n-1$ distinct positive entries. There are exactly $n-1$ possible positive difference values, so the list contains all of them.

Now consider

$$a_{n-1} - a_1 > a_{n-1} - a_2 > \cdots > a_{n-1} - a_{n-2} > 0.$$

These are $n-2$ distinct positive differences. Every one is smaller than $a_n - a_1$, the largest entry of the first list. They must therefore occupy all of its remaining $n-2$ entries:

$$a_n - a_2 > a_n - a_3 > \cdots > a_n - a_{n-1}.$$

Both lists decrease strictly, so their entries agree in order. This gives

$$a_{n-1} - a_i = a_n - a_{i+1} \qquad (1 \le i \le n-2),$$

as asserted by D. Rearranging,

$$a_{i+1} - a_i = a_n - a_{n-1} \qquad (1 \le i \le n-2).$$

The last consecutive gap is already $a_n - a_{n-1} > 0$, so all consecutive gaps are equal. The same argument works for $n = 3$, when each of the two matched lists has one entry.

A confuses the number of gaps with the number of distinct gap values. For $(a_1, a_2, a_3, a_4) = (0, 1, 2, 3)$, the consecutive gaps are $1, 1, 1$, which do not exhaust the first list $3, 2, 1$.

B incorrectly claims that strict ordering alone gives the next smaller entry. For example, with $(a_1, a_2, a_3) = (0, 1, 3)$ the relevant inequality is $1 < 3$, but the next smaller first-list entry is 2, not 1. This example tests B's stated inference from ordering alone; it is not a counterexample to the theorem, since it has three distinct positive differences.

C fails at the smallest first-list entry: subtracting $a_n - a_{n-1}$ from it gives 0, which is not a positive difference. E assumes the arithmetic-progression formula that the proof is meant to establish. Thus only D supplies a valid, non-circular argument.

20. E

Expanding and completing the square gives

$$f(x+1) - f(x) = 3x^2 + 3x + 1 + a = 3\left(x + \frac{1}{2}\right)^2 + a + \frac{1}{4}.$$

At $x = -1/2$, the strict inequality therefore requires $a > -1/4$. This condition also makes the difference positive for every real x, so I must be true.

II need not be true. Take $a = -1/8$, which satisfies the required bound. Then

$$f\left(-\frac{1}{4}\right) = \frac{1}{64} > 0 = f(0),$$

even though $-1/4 < 0$.

For III, if $u \neq v$ and $f(u) = f(v)$, factor the difference to get

$$0 = (u - v)(u^2 + uv + v^2 + a), \qquad u^2 + uv + v^2 = -a < \frac{1}{4}.$$

Also

$$4(u^2 + uv + v^2) - (u - v)^2 = 3(u + v)^2 \geq 0.$$

It follows that $(u - v)^2 < 1$, and hence $|u - v| < 1$. Thus III must be true, including cases in which no two distinct inputs share an output.

Exactly I and III must be true.