1. C

All terms are positive, and

$$\frac{u_{n+1}}{u_n} = \frac{3(n+1)}{4n}.$$

This ratio is greater than 1 when $n < 3$, equal to 1 when $n = 3$, and less than 1 when $n > 3$. Thus $u_1 < u_2 < u_3 = u_4$, and the sequence strictly decreases from u_4 onwards. Its greatest value occurs precisely when

$$\boxed{n = 3 \text{ or } n = 4}.$$

Indeed, $u_3 = u_4 = 27/16$.

2. B

For $f(g(x))$ to be defined, $g(x)$ must be defined and at least 1. Hence

$$x \neq 3, \qquad \frac{1}{x-3} \geq 1.$$

The reciprocal on the left must be positive, so $x > 3$. We may therefore multiply by $x - 3$ without reversing the inequality:

$$1 \geq x - 3, \qquad x \leq 4.$$

The complete condition is $\boxed{3 < x \leq 4}$. The endpoint $x = 3$ is excluded because $g(3)$ is undefined; $x = 4$ is allowed because $f(g(4)) = f(1) = 0$.

3. C

Having exactly three decimal places means that $1000n/240$ is an integer but $100n/240$ is not an integer. Now

$$\frac{1000n}{240} = \frac{25n}{6}, \qquad \frac{100n}{240} = \frac{5n}{12}.$$

Since 25 is coprime to 6, and 5 is coprime to 12, this requires n to be divisible by 6 but not by 12.

Thus the allowed values are

$$6, 18, 30, \ldots, 234.$$

There are $(234 - 6)/12 + 1 = 20$ terms. Their sum is

$$\boxed{\frac{20(6 + 234)}{2} = 2400}.$$

Conversely, each listed n makes $25n/6$ an integer ending in 5, so the resulting decimal does have exactly three places.

4. B

Let the positive edge lengths be a, b and c, labelled so that

$$a^2 + b^2 = 13, \qquad a^2 + c^2 = 20, \qquad b^2 + c^2 = 25.$$

Adding gives $a^2 + b^2 + c^2 = 29$. Subtracting the three original equations gives

$$c^2 = 16, \qquad b^2 = 9, \qquad a^2 = 4.$$

Therefore the edge lengths are 2, 3 and 4, and the volume is

$$\boxed{2 \cdot 3 \cdot 4 = 24}.$$

These lengths reproduce all three given face diagonals.

5. E

Integrating and retaining the lower-limit contribution gives

$$\int_1^x (t^2 - 4t + 3)\,\mathrm{d}t = \frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}.$$

The equation is therefore

$$x^3 - 6x^2 + 9x - 4 = 0.$$

Since $x = 1$ is a root, division by $x - 1$ gives

$$x^3 - 6x^2 + 9x - 4 = (x - 1)(x^2 - 5x + 4) = (x - 1)^2(x - 4).$$

Thus all real solutions are $\boxed{x = 1 \text{ or } x = 4}$. The repeated factor at 1 gives one value of x, and the polynomial factorisation rules out any others.

6. D

Triangles ABD and ACD have equal bases $BD = DC$ on the same line and share the perpendicular height from A. Their areas are equal, so

$$\frac{1}{2}AB \cdot AD \sin 30^\circ = \frac{1}{2}AC \cdot AD \sin 45^\circ.$$

Cancelling the positive common factors gives

$$\frac{AB}{AC} = \frac{\sin 45^\circ}{\sin 30^\circ} = \boxed{\sqrt{2}}.$$

The median need not bisect the angle at A.

7. D

The equation factors as

$$(u - v)(u + v - 5) = 0.$$

Using $v = u^2 - 2$, either

$$u^2 - u - 2 = 0 \quad \text{or} \quad u^2 + u - 7 = 0.$$

The four candidate values of u are

$$2, \quad -1, \quad \frac{-1 + \sqrt{29}}{2}, \quad \frac{-1 - \sqrt{29}}{2}.$$

For a real nonzero x with $x + 1/x = u$, the equation $x^2 - ux + 1 = 0$ must hold. It has two distinct real roots when $u^2 > 4$, one when $u = 2$ or $u = -2$, and none when $u^2 < 4$. Its roots can never be zero.

Thus $u = 2$ contributes the single value $x = 1$, while $u = -1$ contributes none. Since $\sqrt{29} > 5$, the positive surd candidate is greater than 2, and the negative one is less than -2. Each contributes two distinct real values of x.

Different u-values cannot give the same x, because $u = x + 1/x$ is fixed by x. All steps were equivalent, so the total is

$$\boxed{1 + 0 + 2 + 2 = 5}.$$

8. C

The two given coefficients are n and $n(n-1)/2$. Therefore

$$n + \frac{n(n-1)}{2} = 45, \qquad n^2 + n - 90 = (n-9)(n+10) = 0.$$

Since n is positive, $n = 9$. The required coefficient is

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = \boxed{84}.$$

9. A

Let the masses of A and B be $2k$ grams and $3k$ grams. Since volume is mass divided by density,

$$\frac{2k}{8} + \frac{3k}{6} = 150, \qquad \frac{3k}{4} = 150, \qquad k = 200.$$

Thus the mass of A is 400 grams, or $\boxed{0.4 \text{ kg}}$. The corresponding volumes are 50 cm^3 and 100 cm^3, so their sum is indeed 150 cm^3.

10. D

First write

$$y = x^{3/2} + ax^{1/2} - (a+3)x^{-1/2}.$$

Using the power rule gives

$$\frac{dy}{dx} = \frac{3x^2 + ax + a + 3}{2x^{3/2}}.$$

The denominator is positive for $x > 0$. For there to be both a local maximum and a local minimum, the numerator must have two distinct positive roots. Conversely, two such roots give derivative signs $+, -, +$, so they do give the required maximum and minimum.

The product of the roots is $(a+3)/3$ and their sum is $-a/3$. Hence both are positive precisely when they are real and

$$-3 < a < 0.$$

Distinct real roots also require

$$a^2 - 12a - 36 > 0, \qquad a < 6 - 6\sqrt{2} \quad \text{or} \quad a > 6 + 6\sqrt{2}.$$

Combining these conditions gives

$$\boxed{-3 < a < 6 - 6\sqrt{2}}.$$

At $a = -3$, one derivative zero is $x = 0$, outside the domain. At $a = 6 - 6\sqrt{2}$, the numerator has a repeated root and the derivative does not change sign there. Neither endpoint is allowed.

11. D

Put $t = 2^x$, so that $t > 0$ and $4^x = t^2$. Then

$$f(x+1) - f(x) = (4t^2 - 12t) - (t^2 - 6t) = 3t^2 - 6t.$$

Thus

$$3t^2 - 6t = 9, \qquad t^2 - 2t - 3 = (t-3)(t+1) = 0.$$

The root $t = -1$ is impossible because $t = 2^x > 0$. Therefore $t = 3$ and

$$\boxed{x = \log_2 3}.$$

This is unique since 2^x takes each positive value exactly once. At this input, $f(x) = 9 - 18 = -9$ and $f(x+1) = 36 - 36 = 0$, confirming the required difference.

12. B

Since AB is horizontal, the line through C perpendicular to AB is vertical. It passes through H, so $C = (2, h)$ for some h.

The line AH has gradient $3/2$, so the perpendicular line BC has gradient $-2/3$. As it passes through $B = (6, 0)$,

$$y = -\frac{2}{3}(x - 6).$$

At $x = 2$, this gives $h = 8/3$. The triangle therefore has area

$$\frac{1}{2} \cdot 6 \cdot \frac{8}{3} = \boxed{8}.$$

As a check, AC has gradient $4/3$ while BH has gradient $-3/4$, so the third perpendicularity condition also holds.

13. E

The triangle inequalities require $x > 1$. The side x is never the longest. Comparing the other two sides gives two cases.

If $1 < x \leq 2$, the longest side is $x + 1$. Since $\cos 120^\circ = -1/2$, the cosine rule gives

$$(x+1)^2 = x^2 + (2x-1)^2 + x(2x-1).$$

This simplifies to $x(6x - 7) = 0$, giving $x = 7/6$ in the required range.

If $x \geq 2$, the longest side is $2x - 1$. The cosine rule now gives

$$(2x-1)^2 = x^2 + (x+1)^2 + x(x+1).$$

This simplifies to $x(x - 7) = 0$, giving $x = 7$ in the required range. The common boundary $x = 2$ satisfies neither equation.

Both retained values satisfy all strict triangle inequalities, and each equation gives cosine $-1/2$ opposite the longest side. Thus both are attainable, and the requested sum is

$$\frac{7}{6} + 7 = \boxed{\frac{49}{6}}.$$

14. B

The second region lies below the axis, so its signed integral is negative. Thus

$$\int_0^4 f(x)\,dx = 5 - 2 = 3.$$

By linearity,

$$\int_0^4 (2f(x) - 3)\,dx = 2\int_0^4 f(x)\,dx - \int_0^4 3\,dx = 2 \cdot 3 - 3 \cdot 4 = \boxed{-6}.$$

15. D

Independence gives $pq = 1/4$. Thus p and q are the roots of

$$12t^2 - 13t + 3 = (3t - 1)(4t - 3) = 0.$$

As $p > q$, we have $p = 3/4$ and $q = 1/3$.

If B lands tails and there are exactly two heads, both tosses of A must land heads. This has probability

$$p^2(1 - q) = \frac{9}{16} \cdot \frac{2}{3} = \frac{3}{8}.$$

The other way to obtain exactly two heads is for B to land heads and for exactly one toss of A to land heads. There are two orders for the tosses of A, giving probability

$$2p(1 - p)q = 2 \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{8}.$$

Therefore the required conditional probability is

$$\frac{3/8}{3/8 + 1/8} = \boxed{\frac{3}{4}}.$$

16. D

Put $t = x^2 - 4x = (x-2)^2 - 4$. For $1 \leq x < 4$, the complete set of values of t is

$$-4 \leq t < 0.$$

The lower endpoint occurs at $x = 2$. The upper endpoint would require $x = 0$ or $x = 4$, neither of which is allowed.

Now

$$f(x) = t(t+6) = (t+3)^2 - 9.$$

This is at least -9, with equality when $t = -3$, attained at $x = 1$ and $x = 3$. Also $t < 0$ and $t + 6 > 0$, so $f(x) < 0$.

Every value from -9 up to, but not including, 0 occurs: as x runs from 3 towards 4, t runs from -3 towards 0, and $(t+3)^2 - 9$ increases through all those values. Hence

$$\boxed{-9 \leq y < 0}.$$

17. E

Write the other tangent as $y = 2x + c$.

The centre is halfway between two parallel tangents. The line halfway between them is

$$y = 2x + \frac{1+c}{2}.$$

Substituting the centre $(2, -3)$ gives

$$-3 = 4 + \frac{1+c}{2}, \qquad c = -15.$$

Thus the other tangent is $y = 2x - 15$. At its intersection with the x-axis,

$$0 = 2x - 15, \qquad \boxed{x = \frac{15}{2}}.$$

The given tangent does not pass through the centre, so the circle has positive radius and the two parallel tangents are distinct.

18. C

The middle term is $15/5 = 3$. With common difference $d > 0$, the terms are

$$3 - 2d, \quad 3 - d, \quad 3, \quad 3 + d, \quad 3 + 2d.$$

Let the sum of the negative terms be N. Replacing those terms by their absolute values increases the total by $-2N$. Hence

$$19 - 15 = -2N, \qquad N = -2.$$

At least one term is negative, and only the first two can be negative.

If only the first term is negative, then $3 - 2d = -2$, so $d = 5/2$. The second term is then $1/2 > 0$, as required.

If the first two terms are negative, their sum gives $6 - 3d = -2$, so $d = 8/3$. But the second term would then be $3 - 8/3 = 1/3 > 0$, contradicting this case.

Thus the progression is

$$-2, \quad \frac{1}{2}, \quad 3, \quad \frac{11}{2}, \quad 8.$$

It satisfies both given totals, and its largest term is $\boxed{8}$. A zero second term would have $d = 3$ and negative-term sum -3, so it gives no additional solution.

19. C

The sum of the five numbers is $5 \cdot 3 = 15$, and the sum of the ten pairwise products is $10 \cdot 8 = 80$.

On expanding the square of the total, each pairwise product occurs twice. Therefore

$$15^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + 2 \cdot 80.$$

The sum of the five squares is $225 - 160 = 65$, so their mean is

$$\boxed{\frac{65}{5} = 13}.$$

The given summaries are attainable: the list $0, 2, 3, 4, 6$ has sum 15, square-sum 65 and pairwise-product sum $(225 - 65)/2 = 80$.

20. B

Subtracting the second equation from the first at a common solution gives

$$(k-1)x + 1 - k^2 = 0, \qquad (k-1)(x-k-1) = 0.$$

If $k = 1$, both equations become

$$x^2 + x - 2 = (x-1)(x+2) = 0.$$

They then have two common real solutions, so $k = 1$ is not allowed.

For $k \neq 1$, any common solution must be $x = k + 1$. Substitution into the first equation gives

$$(k+1)^2 + k(k+1) - 2 = 0, \qquad 2k^2 + 3k - 1 = 0.$$

This quadratic has two distinct real roots, since its discriminant is $17 > 0$, and neither is 1. For either root, the real number $x = k + 1$ satisfies both original equations. It is the only common solution because their difference is a nonzero linear polynomial in x.

The sum of the two admissible values of k is therefore

$$\boxed{-\frac{3}{2}}.$$