1. D

A counterexample must satisfy the premise and fail the conclusion. At $x = 2$, the premise $x > 1$ is true, but $2^x = x^2 = 4$, so the strict conclusion is false. The values -2 and 1 do not satisfy the premise. At $x = 6$, both the premise and the conclusion hold, since $6 > 1$ and $64 > 36$, so it is not a counterexample. At $x = 5$, the conclusion holds because $32 > 25$.

2. C

Adding the two equations gives

$$a^2 + 2ab + b^2 = 30, \qquad (a+b)^2 = 30.$$

Since a and b are positive, $a + b > 0$. Hence $a + b = \sqrt{30}$.

3. B

For $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = x + 2.$$

Thus the first graph is the line $y = x + 2$ with the point $(2, 4)$ omitted. If $k = 1$, the line $y = kx$ is parallel to it and has no common point. If $k \neq 1$, the corresponding complete lines meet at

$$x = \frac{2}{k - 1}.$$

This intersection is omitted precisely when $2/(k-1) = 2$, which gives $k = 2$. For every other k, the intersection has $x \neq 2$ and is present. Therefore $k = 1$ or $k = 2$.

4. B

For the data set $0, 1, 4, 6, 9$, the mean and median are both 4, so P holds. However, the smallest and largest numbers sum to $9 \neq 8$, so Q fails. Thus P is not sufficient for Q.

For the data set $0, 1, 4, 5, 8$, the median is 4 and the extreme values sum to 8, so Q holds. The mean is $18/5 \neq 4$, so P fails. Thus P is not necessary for Q. Therefore it is neither necessary nor sufficient.

5. E

The contrapositive reverses the implication and negates both its premise and its conclusion. The negation of $w > 6$ or $h > 8$ is $w \leq 6$ and $h \leq 8$. The negation of $d > 10$ is $d \leq 10$. Hence the contrapositive is: if $w \leq 6$ and $h \leq 8$, then $d \leq 10$.

6. C

Subtracting $f(0)$ from the two other given values eliminates c:

$$a(b-1)=2, \qquad a(b^3-1)=14.$$

The first equation shows that $a \neq 0$ and $b \neq 1$. Dividing the second equation by the first and factorising gives

$$\frac{b^3-1}{b-1}=7, \qquad b^2+b+1=7.$$

Hence

$$(b-2)(b+3)=0.$$

Since $b>0$, we have $b=2$. Therefore

$$f(2)-f(1)=ab(b-1)=2b=4,$$

so $f(2)=11$. The constants $a=2$, $b=2$, $c=3$ satisfy all three given values, confirming that the data are attainable.

7. C

Let the radius be r. Tangency to the two parallel bases gives $AD = 2r$. By equal tangent lengths, the two tangent segments from each vertex have equal lengths; adding them around the quadrilateral gives

$$AB + DC = AD + BC.$$

Hence $BC = 13 - 2r$. The horizontal and vertical displacements from B to C are $9 - 4 = 5$ and $2r$, respectively. Pythagoras gives

$$(13 - 2r)^2 = 5^2 + (2r)^2.$$

Therefore $169 - 52r = 25$, and

$$r = \frac{144}{52} = \frac{36}{13}.$$

The resulting height is $72/13$ and $BC = 97/13$, both positive. The top-base tangency point is at distance $r < 4$ from D, so the circle fits the stated configuration.

8. D

To negate the statement, there must be at least one line through the origin for which no circle in the collection has two distinct intersection points with that line.

A straight line can meet a circle at zero, one or two points. Thus this one line must meet every circle at at most one point. The line is fixed while the circles vary; some circles may be missed and others may be tangent to it. This is option D.

9. C

Equal areas on opposite sides of the horizontal line mean that their signed difference is zero. Hence

$$\int_1^2 \left(\frac{x^3+1}{x^2} - k \right) \mathrm{d}x = 0.$$

Simplifying before integrating gives

$$\int_1^2 (x + x^{-2}) \, \mathrm{d}x = \left[\frac{x^2}{2} - \frac{1}{x} \right]_1^2 = \frac{3}{2} - \left(-\frac{1}{2} \right) = 2.$$

The strip has width 1, so the equation is $2 - k = 0$, giving $k = 2$.

This is attainable with nonzero areas on both sides: the curve has height $35/18 < 2$ at $x = 3/2$ and height $9/4 > 2$ at $x = 2$. The signed-area equation is linear in k, so the value is unique.

10. B

Suppose a and b have a common factor $d > 1$. Write $a = du$ and $b = dv$, where u and v are positive integers. Then

$$a^2 + ab + b^2 = d^2(u^2 + uv + v^2).$$

Both factors on the right are integers greater than 1, so this number is not prime. Consequently, Q implies P, proving necessity.

For failure of sufficiency, take $a = 1$ and $b = 4$. Their highest common factor is 1, but

$$a^2 + ab + b^2 = 1 + 4 + 16 = 21$$

is not prime. Therefore P is necessary but not sufficient for Q.

11. D

Allow leading zeros temporarily and consider all five-place strings made from 0 and 1. There are $2^5 = 32$ such strings. One represents zero; the other 31 represent precisely the positive integers in the list with at most five digits. They are the first 31 integers, since the largest is 11111, smaller than every six-digit positive integer.

At any chosen digit position, exactly $2^4 = 16$ strings have a 1: each of the other four positions can independently contain either digit. The all-zero string contributes nothing, so the required sum is

$$16(10000 + 1000 + 100 + 10 + 1) = 16(11111) = 177776.$$

Thus option D is correct. The leading zeros merely give a common way to write the integers; no integer is counted more than once.

12. E

After the transfer, bag B has four balls, so there are six equally likely unordered pairs to choose. Distinguish the transferred ball from the original balls when counting.

If the transferred ball is red, which has probability $3/5$, bag B has three red balls and one blue ball. Three of its six pairs have the same colour; two of those three pairs include the transferred ball.

If the transferred ball is blue, which has probability $2/5$, bag B has two red balls and two blue balls. Two of its six pairs have the same colour; one of those two pairs includes the transferred ball.

Writing S for the event that the pair has the same colour and T for the event that it includes the transferred ball, we obtain

$$\Pr(S) = \frac{3}{5} \cdot \frac{3}{6} + \frac{2}{5} \cdot \frac{2}{6} = \frac{13}{30},$$

$$\Pr(S \text{ and } T) = \frac{3}{5} \cdot \frac{2}{6} + \frac{2}{5} \cdot \frac{1}{6} = \frac{8}{30}.$$

Therefore the required probability is

$$\Pr(T \mid S) = \frac{8/30}{13/30} = \frac{8}{13}.$$

13. D

Lines I and II are valid algebraic deductions. Line III is also valid because both sides of the inequality being squared are nonnegative.

The first error is in line IV. The bound $ab \leq 1$ does not imply $|ab| \leq 1$, because ab need not be nonnegative. For example, $a = -1$ and $b = 3$ satisfy $a + b = 2$ and $ab = -3 \leq 1$, but $a^2b^2 = 9 > 1$.

Line V would follow from the bounds in III and IV, but line IV has not been established. The fact that the final inequality is true does not make this proof valid. Indeed, a valid alternative is

$$a^4 + b^4 \geq \frac{(a^2 + b^2)^2}{2} \geq 2,$$

where the first inequality follows by expanding $(a^2 - b^2)^2 \geq 0$.

14. A

Suppose P holds, and let r be either root. If $|r| \geq 1$, then the root equation gives

$$r^2 = |ar + b| \leq |a|\,|r| + |b| \leq (|a| + |b|)|r| < |r| \leq r^2,$$

a contradiction. Thus $|r| < 1$ for both roots, proving that P is sufficient for Q.

It is not necessary. Consider

$$\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right) = x^2 - \frac{5}{4}x + \frac{3}{8}.$$

Its two distinct real roots both lie strictly between -1 and 1, but

$$|a| + |b| = \frac{5}{4} + \frac{3}{8} = \frac{13}{8} > 1.$$

Thus Q can hold without P, so P is sufficient but not necessary for Q.

15. C

Write $s = \sin\theta$ and $c = \cos\theta$. The left-hand side is strictly positive, so a solution must have $s + c > 0$. Squaring gives

$$2 + 2\sqrt{1 - s^2} = 2(s + c)^2 = 2 + 4sc,$$

so $|c| = 2sc$.

If $c = 0$, the candidates are $\theta = \pi/2$ and $3\pi/2$. Only $\pi/2$ has $s + c > 0$.

If $c > 0$, division by c gives $s = 1/2$. The only candidate with $c > 0$ is $\theta = \pi/6$, and $s + c > 0$ there.

If $c < 0$, division by c gives $s = -1/2$. The only candidate with $c < 0$ is $\theta = 7\pi/6$, where $s + c < 0$, so it is rejected.

The two retained candidates satisfy the squared equation and have a positive right-hand side, so they also satisfy the original equation. Their sum is

$$\frac{\pi}{6} + \frac{\pi}{2} = \boxed{\frac{2\pi}{3}}.$$

16. D

If P holds, then $x = y > 0$, and

$$\log_2(x+y) = \log_2(2x) = 1 + \log_2 x = 1 + \frac{\log_2 x + \log_2 y}{2}.$$

Thus P is sufficient for Q.

Conversely, if Q holds, multiply by 2 and use the logarithm laws:

$$\log_2\big((x+y)^2\big) = \log_2(4xy).$$

Hence $(x+y)^2 = 4xy$, which gives $(x-y)^2 = 0$ and therefore $x = y$. Thus P is also necessary for Q.

All logarithm arguments are positive under the stated domain. The answer is both necessary and sufficient.

17. B

The proposed replacement must improve the product for either order of the two interior numbers b, c. Pairing the extremes gives

$$\begin{aligned}(a+d)(b+c) - (a+b)(c+d) &= ab + cd - ad - bc \\ &= (d-b)(c-a) > 0.\end{aligned}$$

Every other pair sum is positive and unchanged, so the entire product strictly increases. This contradicts the maximality of the chosen pairing, establishing the missing argument in option B.

The replacement in A has change $(c-b)(d-a)$, whose sign is not fixed. C gives the wrong factorisation for the stated replacement. D incorrectly claims that both new sums are larger; in particular $b + c < c + d$. E confuses an unchanged total with an increase in product.

A maximum exists because there are finitely many pairings. Once the smallest and largest remaining numbers are paired, the same exchange argument can be applied successively until no numbers remain. No infinite process or assumption about the order of b, c is needed.

18. C

Let the original list have n entries $x_1, \ldots, x_n$, where $n \geq 1$. The totals in the two new lists are $13n$ and $5n$.

For every original entry x,

$$(x-2)^2 - (x-6)^2 = (x^2 - 4x + 4) - (x^2 - 12x + 36) = 8x - 32.$$

Summing this identity over the original list gives

$$13n - 5n = 8(x_1 + \cdots + x_n) - 32n.$$

Therefore $x_1 + \cdots + x_n = 5n$, so the original mean is $\boxed{5}$.

The data are attainable: the original list $3, 7$ gives first-list entries $1, 25$, of mean 13, and second-list entries $9, 1$, of mean 5. The same identity fixes the original mean uniquely for every non-empty list satisfying the conditions.

19. F

Since a and b have no common prime factor, each prime factor of ab occurs wholly in a or wholly in b. The exponents in the perfect square ab are all even. Thus the prime exponents in each of a and b are all even, so I is true.

Write $a = u^2$ and $b = v^2$. They cannot both be even because their highest common factor is 1. If they were both odd, then both u and v would be odd. An odd square has remainder 1 on division by 4, so $a + b$ would have remainder 2. But a square has remainder 0 or 1 on division by 4. This contradicts the hypothesis that $a+b$ is a square. Exactly one of a, b is therefore even, and II is true.

For III, take $a = 9$ and $b = 16$. Their highest common factor is 1, $ab = 144 = 12^2$, and $a+b = 25 = 5^2$, but neither 9 nor 16 is divisible by 5. Thus III is false.

Only I and II must be true.

20. A

By similarity, when the water has depth z, its surface radius is $3z/5$. Hence its volume is

$$V(z) = \frac{1}{3}\pi\left(\frac{3z}{5}\right)^2 z = \frac{3\pi}{25}z^3.$$

Let h be the depth at the end of the first interval. The three successive depths are $h-2$, h, and $h+1$. Equal times and constant volume rate imply equal volume increases, so

$$h^3 - (h-2)^3 = (h+1)^3 - h^3.$$

Expanding and cancelling the cubic terms gives

$$3h^2 - 15h + 7 = 0, \qquad h = \frac{15 \pm \sqrt{141}}{6}.$$

Initially there is water, so $h > 2$. Since $11 < \sqrt{141} < 12$, the minus root is less than 1 and is inadmissible. The plus root lies between $13/3$ and $9/2$, so $h - 2 > 0$ and $h + 1 < 10$ as required.

The required depth is therefore $\boxed{\dfrac{15+\sqrt{141}}{6}}$ centimetres. At this depth the two positive volume increments are equal, so a positive constant filling rate realizes the stated intervals.