1. D

The difference of two cubes factorises as

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4).$$

Since $x \neq 2$, the expression is

$$x^2 + 2x + 4 - x(x + 2) = \boxed{4}.$$

2. C

For each positive integer n, take the contrapositive of the given implication. The negation of $u_{n+1} > 0$ is $u_{n+1} \leq 0$, and the negation of $u_n > 0$ is $u_n \leq 0$.

Thus the equivalent property is: for every positive integer n, if $u_{n+1} \leq 0$, then $u_n \leq 0$. This is option C.

A and B reverse the required implication. D mishandles zero, and E weakens the conclusion.

3. C

The denominator must be nonzero, so $\sin\theta \neq 1$. Put $s = \sin\theta$. Then $\cos\theta = 2(1-s)$, and the identity $\sin^2\theta + \cos^2\theta = 1$ gives

$$s^2 + 4(1-s)^2 = 1, \qquad 5s^2 - 8s + 3 = (5s-3)(s-1) = 0.$$

The root $s = 1$ is excluded by the original denominator. Thus $s = 3/5$. This is attainable with $\cos\theta = 4/5$, for which the original quotient is 2. Hence the answer is $\boxed{3/5}$.

4. B

Let C be the angle opposite the side of length c. By the cosine rule, C is acute precisely when

$$a^2 + b^2 > c^2.$$

Since $(a - b)^2 \geq 0$,

$$a^2 + b^2 \geq \frac{(a + b)^2}{2}.$$

If P holds, the last quantity is greater than c^2. Thus P is sufficient for Q.

For the converse, take $a = 1$ and $b = c = 3$. These are the sides of a nondegenerate triangle, and $a^2 + b^2 = 10 > 9 = c^2$, so Q holds. However, $a + b = 4 < 3\sqrt{2}$, since $16 < 18$. Thus P fails and is not necessary. The answer is B.

5. D

A counterexample must satisfy the premise and fail the conclusion. For $m = 3$ and $n = 8$,

$$m + n = 11 \quad \text{is prime,} \qquad mn + 1 = 25 = 5^2 \quad \text{is not prime.}$$

Thus D is a counterexample. In A and C both the premise and conclusion hold. In B and E the premise fails, since the respective sums are 12 and 9.

6. D

Let D be the event that the first and third balls have different colours, and let R be the event that the second ball is red.

The first and third balls have the same colour only if both are red. Therefore

$$\Pr(D) = 1 - \frac{2}{4} \cdot \frac{1}{3} = \frac{5}{6}.$$

If the second ball is red, the other three balls consist of one red, one blue and one green, so the first and third balls necessarily have different colours. Hence R implies D, and

$$\Pr(R \text{ and } D) = \Pr(R) = \frac{2}{4} = \frac{1}{2}.$$

The required conditional probability is

$$\frac{\Pr(R \text{ and } D)}{\Pr(D)} = \frac{1/2}{5/6} = \boxed{\frac{3}{5}}.$$

7. E

Adding 1 to the recurrence gives

$$u_{n+1} + 1 = 3(u_n + 1).$$

Thus $u_n + 1$ is a geometric sequence with first term 2 and common ratio 3. Its reciprocals form a geometric sequence with first term $1/2$ and common ratio $1/3$.

The required sum of five terms is

$$\frac{1}{2}\frac{1-(1/3)^5}{1-1/3} = \frac{3}{4}\left(1 - \frac{1}{243}\right) = \boxed{\frac{121}{162}}.$$

8. D

The original statement says that every row has at least one entry which is strictly greater than all the other entries in its column. Its negation says that there is at least one row with no such entry.

Fix this row. In each column, failure to be strictly greater than every other entry means that at least one other entry in that column is greater than or equal to its entry. The competing row need not be the same in different columns, and equality is enough to prevent a strict maximum. This is precisely option D.

9. C

To disprove sufficiency, take $a = \sqrt{2}$ and $b = 2 - \sqrt{2}$, both positive. Then $a + b = 2$ is rational, but

$$a^2 + b^2 = 8 - 4\sqrt{2}$$

is irrational. Thus P can hold while Q fails.

To disprove necessity, take $a = b = \sqrt{2}$. Then $a^2 + b^2 = 4$ is rational, but $a + b = 2\sqrt{2}$ is irrational. Thus Q can hold while P fails.

Therefore P is neither necessary nor sufficient for Q.

10. B

The line AC is perpendicular to a line of gradient 2, so it has gradient $-1/2$. Since it passes through $A = (0, 4)$, its equation is

$$y = -\frac{1}{2}x + 4.$$

The midpoint M lies on both AC and BM. Thus

$$x + 1 = -\frac{1}{2}x + 4, \qquad x = 2, \qquad y = 3.$$

If $C = (u, v)$, the midpoint formula gives

$$\left(\frac{u}{2}, \frac{4+v}{2}\right) = (2, 3).$$

Hence $u = 4$ and $v = 2$, so $C = (4, 2)$.

11. E

Factor the logarithm arguments as $(x-1)(x-2)$ and $(x-2)(x-3)$. Both are positive exactly when $x < 1$ or $x > 3$.

On this domain, combining the logarithms gives

$$(x-1)(x-2)^2(x-3) = 2.$$

Put $u = x - 2$. Then

$$u^2(u^2-1) = 2, \qquad (u^2-2)(u^2+1) = 0.$$

For real u, only $u^2 = 2$ is possible. Thus $x = 2 - \sqrt{2}$ or $x = 2 + \sqrt{2}$. Since $\sqrt{2} > 1$, these satisfy $x < 1$ and $x > 3$ respectively, so both are valid and distinct.

The required sum is

$$(2-\sqrt{2})^2 + (2+\sqrt{2})^2 = 12.$$

12. A

First suppose P holds. For every real x,

$$(a^2+b^2)(x^2+1)-(ax+b)^2=(a-bx)^2\geq 0.$$

Therefore

$$(ax+b)^2\leq (a^2+b^2)(x^2+1)\leq x^2+1.$$

Taking nonnegative square roots gives $|ax+b|\leq\sqrt{x^2+1}$, which implies Q. Thus P is sufficient.

Conversely, suppose Q holds and put $x=a/b$, which is permitted because $b>0$. Writing $s=a^2+b^2>0$, the resulting inequality is

$$\frac{s}{b}\leq\sqrt{\frac{s}{b^2}}=\frac{\sqrt{s}}{b}.$$

Thus $s\leq\sqrt{s}$, so $\sqrt{s}\leq 1$ and $s\leq 1$. This is P, proving necessity.

Hence P is both necessary and sufficient for Q.

13. D

Write $f(x) = ax^2 + bx + c$. The exact area is

$$\int_0^2 f(x)\,dx = \frac{8a}{3} + 2b + 2c = 9.$$

Each trapezium has width 1, so the estimate is

$$\frac{f(0) + 2f(1) + f(2)}{2} = 3a + 2b + 2c = 10.$$

Subtracting the exact area from the estimate gives $a/3 = 1$, hence $a = 3$. Since $f'(x) = 2ax + b$,

$$f'(2) - f'(0) = 4a = 12.$$

14. C

Lines I and II are valid: one may assume a positive solution when arguing by contradiction, and division by that positive number is permitted.

The first error is in line III. The equation $x^2 - 1 = 6x^{-1}$ is only assumed to hold at a solution. It is not an identity holding for all x in a neighbourhood, so differentiating its two sides does not preserve the solutions. The derivative calculations themselves are correct, but their equality has not been justified.

Indeed, $x = 2$ satisfies $x^3 - x = 6$, confirming that the claimed conclusion is false.

15. C

Let the side length be $s > 0$, and choose coordinates

$$A = (0,0), \quad B = (s,0), \quad D = (0,s), \quad P = (x,y).$$

Since P is strictly inside the square, $0 < x < s$ and $0 < y < s$. The three given distances yield

$$x^2 + y^2 = 5, \qquad (x-s)^2 + y^2 = 13, \qquad x^2 + (y-s)^2 = 17.$$

Subtracting the first equation from each of the others gives

$$s^2 - 2sx = 8, \qquad s^2 - 2sy = 12.$$

Hence

$$x = \frac{s^2 - 8}{2s}, \qquad y = \frac{s^2 - 12}{2s}.$$

Put $u = s^2$, the required area. Substitution into $x^2 + y^2 = 5$ gives

$$(u-8)^2 + (u-12)^2 = 20u,$$

so

$$u^2 - 30u + 104 = (u-4)(u-26) = 0.$$

The choice $u = 4$ gives $s = 2$ and $P = (-1,-2)$, outside the square, so it is not allowed. For $u = 26$,

$$s = \sqrt{26}, \qquad x = \frac{9}{\sqrt{26}}, \qquad y = \frac{7}{\sqrt{26}},$$

which satisfy $0 < x < s$ and $0 < y < s$. Their squared distances to A, B and D are respectively

$$\frac{9^2 + 7^2}{26} = 5, \qquad \frac{17^2 + 7^2}{26} = 13, \qquad \frac{9^2 + 19^2}{26} = 17.$$

Thus the unique possible area is $\boxed{26}$.

16. B

Independence gives

$$\Pr(\text{exactly one head}) = p(1-q) + (1-p)q.$$

If $p = q$, this probability is

$$2p(1-p) = \frac{1}{2} - 2\left(p - \frac{1}{2}\right)^2 \leq \frac{1}{2}.$$

Consequently Q cannot hold when P is false: Q implies $p \neq q$, so P is necessary.

P is not sufficient. For example, if $p = 1/4$ and $q = 1/2$, then $p \neq q$ but

$$p(1-q) + (1-p)q = \frac{1}{8} + \frac{3}{8} = \frac{1}{2},$$

which is not greater than $1/2$. Thus P is necessary but not sufficient for Q.

17. E

The useful factorisation is

$$n^4 + n^2 + 1 = (n^2 + 1)^2 - n^2 = (n^2 - n + 1)(n^2 + n + 1).$$

The two quadratic factors differ by $2n$, so

$$\frac{2n}{n^4 + n^2 + 1} = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}.$$

Also,

$$n^2 + n + 1 = (n + 1)^2 - (n + 1) + 1.$$

Thus consecutive terms cancel in the sum:

$$\begin{aligned}\sum_{n=1}^{20} \frac{2n}{n^4 + n^2 + 1} &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{381} - \frac{1}{421}\right) \\ &= 1 - \frac{1}{421} = \boxed{\frac{420}{421}}.\end{aligned}$$

All denominators are positive, so every displayed manipulation is valid.

18. C

Replacing x by $x+1$ gives

$$f(x+1)+f(x+2)=2x+3.$$

Subtracting the original identity yields $f(x+2)-f(x)=2$, so I must be true.

Repeatedly applying I, for each positive integer n we obtain

$$f(2n)=f(0)+2n.$$

These values exceed any prescribed real number once n is large enough. Thus II must also be true.

III need not be true. For example, let

$$f(x)=x+\cos(\pi x).$$

Since $\cos(\pi x+\pi)=-\cos(\pi x)$, this function satisfies the required identity for every real x. But $f(0)=1$ and $f(1)=0$, so $0<1$ does not give $f(0)<f(1)$. Therefore only I and II must be true.

19. A

Let T be the sum of the ten affected temperatures measured in degrees Celsius. The correct sum of all thirty Celsius values is $30 \cdot 18 = 540$.

Replacing those ten entries by Fahrenheit values changes their sum from T to

$$\frac{9}{5}T + 10 \cdot 32.$$

The resulting increase in the total is therefore $\frac{4}{5}T + 320$. The reported mean shows that the increase was

$$30(30 - 18) = 360.$$

Hence

$$\frac{4}{5}T + 320 = 360, \qquad T = 50.$$

The required mean is $T/10 = \boxed{5^\circ\mathrm{C}}$.

The data are consistent: the ten affected Celsius values may all be 5, and the other twenty may all be 24.5. Their true mean is 18, while the incorrectly entered group consists of ten values of 41, giving the reported mean 30.

20. F

Choose the last occurrence of the minimum among $S_0, \ldots, S_{n-1}$. Since $S_0 = 0$, we have $S_j \leq 0$.

If $j < k < n$, minimality gives $S_k \geq S_j$. Equality is impossible because j was the last index attaining this minimum. Therefore $S_k - S_j > 0$, justifying line 2.

Since $S_n > 0$ and $S_j \leq 0$, we have $S_n - S_j > 0$, justifying line 3.

For a wrapped initial sum, direct addition gives

$$(a_{j+1} + \cdots + a_n) + (a_1 + \cdots + a_k) = S_n - S_j + S_k.$$

Here $S_k \geq S_j$, so this sum is at least $S_n > 0$. This proves line 4 and identifies F.

Choosing the first minimum need not work: for $(a_1, a_2) = (0, 1)$ it chooses $j = 0$, leaving first sum 0. Choosing a maximum fails for $(3, -2)$, since the rearrangement begins with -2. Choosing $j = 0$ always fails for $(-1, 2)$. Options C and D use incorrect expressions for the wrapped sum; for example, with $(-1, -1, 3)$ the chosen index is $j = 2$, and the wrapped sum ending at a_1 is 2, whereas those expressions give 0 and 0, respectively. Thus F is the only valid completion.