1. C

Expanding and subtracting gives

$$f(a+1) - f(a) = 4a - 1.$$

Thus $4a - 1 = 9$, so $a = 5/2$. Similarly,

$$f(a-1) - f(a) = -4a + 5 = -10 + 5 = \boxed{-5}.$$

Alternatively, consecutive forward differences of this quadratic differ by 4. Hence $f(a) - f(a-1) = 9 - 4 = 5$, and the requested difference has the opposite sign.

2. C

After the drink is poured out, 600 mL remains, with the same concentrate-to-water ratio 1 : 4. It contains

$$\frac{1}{5} \times 600 = 120\,\text{mL}$$

of concentrate and 480 mL of water.

The amount of concentrate does not change when water is added. A ratio of 1 : 9 therefore requires $9 \times 120 = 1080$ mL of water. The amount added is

$$1080 - 480 = \boxed{600\,\text{mL}}.$$

3. C

Let M be the midpoint of chord AB. The line OM is perpendicular to AB, so its line has gradient $4/3$ and equation $y = 4x/3$.

Solving this with $3x + 4y = 15$ gives

$$M = \left(\frac{9}{5}, \frac{12}{5}\right), \qquad OM = \sqrt{\left(\frac{9}{5}\right)^2 + \left(\frac{12}{5}\right)^2} = 3.$$

The circle has radius 5, so Pythagoras in triangle OMA gives

$$AM = \sqrt{5^2 - 3^2} = 4.$$

Thus $AB = 8$, and the required area is

$$\frac{1}{2} \times 8 \times 3 = \boxed{12}.$$

4. B

The terms of $(1+x)^5$ through degree 3 are

$$1 + 5x + 10x^2 + 10x^3.$$

After multiplication by $1 + ax$, the coefficient of x^2 is $10 + 5a$. Thus $a = -2$.

The coefficient of x^3 is then

$$10 + 10a = 10 - 20 = \boxed{-10}.$$

5. D

Put $u = x^2 - 3x$. The equation becomes

$$u^2 = 2u + 8, \qquad (u-4)(u+2) = 0.$$

If $u = 4$, then

$$x^2 - 3x - 4 = (x-4)(x+1) = 0,$$

so $x = 4$ or $x = -1$.

If $u = -2$, then

$$x^2 - 3x + 2 = (x-1)(x-2) = 0,$$

so $x = 1$ or $x = 2$.

These four values all satisfy the original equation, and the substitution introduces no other restrictions. The positive solutions are 1, 2 and 4, whose sum is $\boxed{7}$.

6. D

Write $OT = h$ and $OA = d$, with lengths measured in metres. The two right-angled triangles give

$$h = d\tan 60^\circ = \sqrt{3}\,d, \qquad h = (d+20)\tan 30^\circ = \frac{d+20}{\sqrt{3}}.$$

Equating these expressions gives $3d = d + 20$, so $d = 10$. Therefore

$$OT = \boxed{10\sqrt{3}\,\text{m}}.$$

7. B

Since f' is quadratic and $f'(0) = f'(4) = 9$, the quadratic $f'(x) - 9$ has roots 0 and 4. Therefore, for some nonzero constant k,

$$f'(x) = 9 + kx(x - 4).$$

The Fundamental Theorem of Calculus gives

$$4 = f(4) - f(0) = \int_0^4 (9 + kx^2 - 4kx)\,\mathrm{d}x = 36 - \frac{32k}{3}.$$

It follows that $k = 3$, and hence

$$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3).$$

The stationary points have x-coordinates 1 and 3. Both are genuine maxima or minima because the derivative changes sign at each. Their positive separation is $\boxed{2}$.

8. C

There are four even numbers, so there are $4 \cdot 3/2 = 6$ pairs in which both numbers are even. Every such pair has a product divisible by 4.

If just one number is even, the product is divisible by 4 precisely when that number is 4 or 8. Each can be paired with any of the four odd numbers, giving $2 \cdot 4 = 8$ further pairs. Two odd numbers cannot have a product divisible by 4.

Thus the condition leaves $6 + 8 = 14$ equally likely pairs, of which 6 have both numbers even. The required probability is

$$\frac{6}{14} = \boxed{\frac{3}{7}}.$$

9. B

The curve is on or above the x-axis because its numerator is a square and $x > 0$. Expanding before integrating gives

$$\frac{(x-1)^2}{\sqrt{x}} = x^{3/2} - 2x^{1/2} + x^{-1/2}.$$

Hence the area is

$$\begin{aligned}\int_1^4 \left(x^{3/2} - 2x^{1/2} + x^{-1/2}\right)\,\mathrm{d}x &= \left[\frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2} + 2x^{1/2}\right]_1^4 \\ &= \left(\frac{64}{5} - \frac{32}{3} + 4\right) - \left(\frac{2}{5} - \frac{4}{3} + 2\right) \\ &= \frac{92}{15} - \frac{16}{15} = \boxed{\frac{76}{15}}.\end{aligned}$$

10. D

The right-angled triangles ACD and CBD are similar: each has a right angle at D, and $\angle ACD = \angle CBD$ because both are complementary to $\angle CAD$.

Corresponding sides therefore give

$$\frac{AD}{CD} = \frac{CD}{DB}, \qquad CD^2 = AD \cdot DB = 9 \cdot 16 = 144.$$

As CD is a length, $CD = 12$. Also, $AB = 9 + 16 = 25$, so the required area is

$$\frac{1}{2} AB \cdot CD = \frac{1}{2} \cdot 25 \cdot 12 = \boxed{150}.$$

11. E

Suppose the roots are r and s, with $0 < r < s < 4$. Comparing coefficients in

$$x^2 - kx + k = (x - r)(x - s)$$

gives $rs = k > 0$. The discriminant is

$$k^2 - 4k = k(k - 4) > 0,$$

so $k > 4$. Since 4 lies strictly to the right of both roots,

$$16 - 3k = (4 - r)(4 - s) > 0,$$

which gives $k < 16/3$. These conditions are necessary.

Conversely, if $4 < k < 16/3$, the discriminant is positive. The two roots have positive product k and positive sum k, so both are positive. The vertex is at

$$x = \frac{k}{2} < \frac{8}{3} < 4.$$

The value at $x = 4$ is positive. Since the upward-opening quadratic is negative strictly between its roots and its vertex lies between them, 4 must lie strictly to the right of both roots. Thus both roots are less than 4.

At $k = 4$ the root is repeated; at $k = 16/3$ one root is 4. Both endpoints are excluded, giving

$$\boxed{4 < k < \frac{16}{3}}.$$

12. A

Let the first term be a and the common ratio be r. Since the terms are positive, $a > 0$ and $r > 0$.

The second three-term block is r^3 times the first, so

$$r^3 \cdot 21 = 168, \qquad r^3 = 8, \qquad r = 2.$$

It follows that

$$a(1 + 2 + 4) = 21, \qquad a = 3.$$

The first three terms are therefore $3, 6, 12$, and the required value is

$$3^2 + 6^2 + 12^2 = 9 + 36 + 144 = \boxed{189}.$$

13. F

Squaring the equality of the two sine–cosine sums and using $\sin^2\theta + \cos^2\theta = 1$ shows that

$$\sin\alpha\cos\alpha = \sin\beta\cos\beta.$$

Thus the pairs $(\sin\alpha, \cos\alpha)$ and $(\sin\beta, \cos\beta)$ have the same sum and product. They are therefore the same unordered pair, since each pair consists of the roots of the same quadratic.

Sine is strictly increasing for $0 < \theta < \pi/2$, and $\alpha < \beta$, so $\sin\alpha \neq \sin\beta$. The pairs must be swapped:

$$\sin\beta = \cos\alpha, \qquad \cos\beta = \sin\alpha.$$

Consequently,

$$4 = \tan\alpha + \tan\beta = \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\sin\alpha\cos\alpha},$$

so $\sin\alpha\cos\alpha = 1/4$. Hence

$$(\sin\beta - \sin\alpha)^2 = (\cos\alpha - \sin\alpha)^2 = 1 - 2\sin\alpha\cos\alpha = \frac{1}{2}.$$

The required difference is positive because $\alpha < \beta$. Therefore it equals $\boxed{\sqrt{2}/2}$.

14. B

If the actual dimensions are l cm and w cm, their upper bounds are 12.5 and 8.5. Thus the area cannot exceed

$$12.5 \cdot 8.5 = \frac{25 \cdot 17}{4} = \frac{425}{4} = 106.25 \text{ cm}^2,$$

so it is certainly less than 107 cm^2.

The dimensions 12.49 cm and 8.49 cm both round as stated, and their product is 106.0401 cm^2, which is greater than 106 cm^2. Thus no integer at most 106 is a guaranteed strict upper bound.

The least possible integer is therefore $\boxed{107}$.

15. D

Completing the square gives

$$f(x) = 1 + \frac{3}{(x+1)^2 + 1}.$$

Since $-3 < x < 0$, we have $-2 < x + 1 < 1$. Thus $(x+1)^2$ takes every value satisfying

$$0 \leq (x+1)^2 < 4.$$

The value 0 is attained at $x = -1$, whereas 4 is not attained. Therefore

$$\boxed{\frac{8}{5} < f(x) \leq 4}.$$

Every value between these bounds occurs: as x goes from just above -3 to -1, $(x+1)^2$ decreases through all values from just below 4 to 0.

16. B

Put $\theta = \angle ABC$. Opposite angles of a cyclic quadrilateral are supplementary, so

$$\angle ADC = 180^\circ - \theta.$$

Apply the cosine rule to the common diagonal AC in triangles ABC and ADC:

$$AC^2 = 3^2 + 4^2 - 2(3)(4)\cos\theta = 5^2 + 6^2 + 2(5)(6)\cos\theta.$$

Hence $25 - 24\cos\theta = 61 + 60\cos\theta$, giving

$$\cos\theta = -\frac{3}{7}, \qquad \sin\theta = \sqrt{1 - \frac{9}{49}} = \frac{2\sqrt{10}}{7}.$$

The positive square root is required because $0 < \theta < 180^\circ$.

The two triangles have the same sine of their included angles, so their areas add to

$$\frac{1}{2}(3 \cdot 4 + 5 \cdot 6)\sin\theta = 21 \cdot \frac{2\sqrt{10}}{7} = \boxed{6\sqrt{10}}.$$

17. D

Frequency equals class width multiplied by frequency density. The three frequencies are therefore 20, $20k$ and 30, and their class midpoints are 5, 20 and 35.

The estimated mean gives

$$\frac{20 \cdot 5 + 20k \cdot 20 + 30 \cdot 35}{50 + 20k} = 22.$$

Thus $1150 + 400k = 1100 + 440k$, so $k = 5/4$. The frequencies are 20, 25 and 30, with total 75.

Half the total frequency is 37.5, so the median lies in the middle class. Linear interpolation gives

$$10 + \frac{37.5 - 20}{25} \cdot 20 = 10 + 14 = \boxed{24 \text{ minutes}}.$$

18. C

The average rate of increase over the first 3 minutes is

$$\frac{V(3) - V(0)}{3} = \frac{30 - 3}{3} = 9$$

litres per minute.

The instantaneous rate is

$$V'(t) = 3t^2.$$

Therefore $3a^2 = 9$, giving $a = \sqrt{3}$ or $a = -\sqrt{3}$. Since a is a time in the first 3 minutes, only

$$\boxed{a = \sqrt{3}}$$

is admissible.

19. F

By the binomial theorem,

$$a_r = \binom{9}{r} = \binom{9}{9-r} = a_{9-r}.$$

The coefficient of x^8 in the product of two copies of $(1+x)^9$ is

$$a_0a_8 + a_1a_7 + \cdots + a_8a_0.$$

For $0 \le r \le 8$, symmetry gives $a_{8-r} = a_{r+1}$. Hence this coefficient is exactly

$$a_0a_1 + a_1a_2 + \cdots + a_8a_9.$$

But the product is $(1+x)^{18}$, so the required value is

$$\boxed{\binom{18}{8}}.$$

20. C

Write $s = x + y$ and $p = xy$. From the square sum,

$$s^2 - 2p = 13, \qquad p = \frac{s^2 - 13}{2}.$$

Also, $x^3 + y^3 = s^3 - 3ps = 35$. Substituting for p gives

$$s^3 - 39s + 70 = 0, \qquad (s-2)(s-5)(s+7) = 0.$$

Thus s is 2, 5 or -7.

For x and y to be real, the quadratic $z^2 - sz + p = 0$ must have nonnegative discriminant:

$$s^2 - 4p = 26 - s^2 \geq 0.$$

This excludes $s = -7$ and permits both $s = 2$ and $s = 5$. These give, respectively,

$$p = -\frac{9}{2} \qquad \text{and} \qquad p = 6.$$

Conversely, each permitted pair (s, p) has real roots and satisfies the two original power-sum equations, so both product values occur. Their sum is

$$-\frac{9}{2} + 6 = \boxed{\frac{3}{2}}.$$