1. The real numbers a and b satisfy

$$a + b = 6, \qquad ab = 7.$$

What is the value of

$$\frac{a}{b} + \frac{b}{a}\,?$$

A $\frac{6}{7}$

B $\frac{8}{7}$

C $\frac{22}{7}$

D $\frac{29}{7}$

E $\frac{36}{7}$

2. Consider the following claim about nonzero real numbers a and b:

$$\text{If } a < b, \text{ then } \frac{1}{a} > \frac{1}{b}.$$

Which of the following is a counterexample to this claim?

A $a = 1, \quad b = 3$

B $a = -4, \quad b = -2$

C $a = 2, \quad b = -1$

D $a = -1, \quad b = -3$

E $a = -2, \quad b = 3$

3. The graphs

$$y = \sqrt{x+4} \qquad \text{and} \qquad y = 2 - x$$

intersect at P. What are the coordinates of P?

A $(-4, 0)$

B $(0, 2)$

C $(2, 0)$

D $(5, -3)$

E $(5, 3)$

4. Let $f(x) = x^2 + px + q$, where p and q are real constants, and suppose that $f(x) = 0$ has two distinct real roots. Consider the statements

$$\begin{aligned} P: &\quad f(1) > 0, \\ Q: &\quad \text{both roots of } f(x) = 0 \text{ are greater than 1.} \end{aligned}$$

Which of the following is true?

A P is sufficient but not necessary for Q.

B P is both necessary and sufficient for Q.

C P is neither necessary nor sufficient for Q.

D P is necessary but not sufficient for Q.

5. For some positive real number x, the two roots of

$$t^2 - 6t + k = 0$$

are $\log_2 x$ and $\log_2(x + 12)$. What is the value of k?

A 8

B -8

C 4

D 6

E 16

F 2

6. A function f is defined for all real numbers. Consider the statement:

$$\text{For every real } x, \text{ if } x < 0 \text{ or } x > 3, \text{ then } f(x) > 1.$$

Which of the following statements is logically equivalent to it? Each option applies to every real x.

A If $0 \leq x \leq 3$, then $f(x) \leq 1$.

B If $f(x) > 1$, then $x < 0$ or $x > 3$.

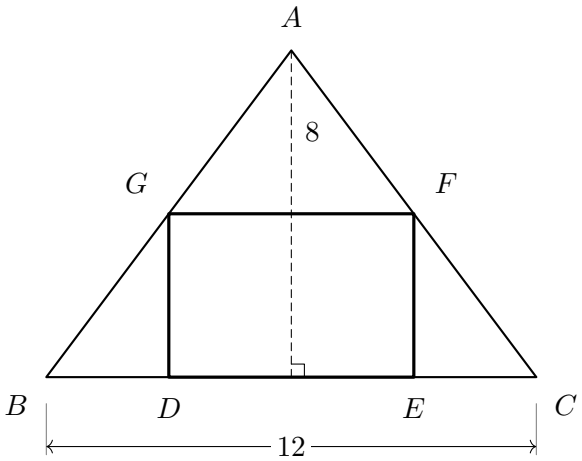
C If $f(x) \leq 1$, then $0 \leq x \leq 3$.

D If $f(x) < 1$, then $0 < x < 3$.

E If $f(x) \leq 1$, then $x \leq 0$ or $x \geq 3$.

7. Triangle ABC has base $BC = 12$ and perpendicular height 8. The vertices D and E of rectangle $DEFG$ lie on BC, with G on AB and F on AC, as shown.

The perimeter of the rectangle is 20. What is its area?



A 12

B 16

C 20

D 24

E 32

F 48

8. Let a and b be positive real numbers. Consider the statements

P: $a + b \leq 4$.

Q: $(a + 1)(b + 1) \leq 9$.

Which of the following is correct?

A P is necessary but not sufficient for Q

B P is both necessary and sufficient for Q

C P is neither necessary nor sufficient for Q

D P is sufficient but not necessary for Q

9. What is the area of the region consisting of all points (x, y) satisfying

$$x \geq 0, \qquad y \geq 0, \qquad y \leq x^2, \qquad x + y \leq 2\,?$$

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{5}{6}$

D $\frac{7}{6}$

E 2

10. Let a and b be distinct positive real numbers satisfying

$$a + \frac{1}{a} = b + \frac{1}{b}.$$

A student gives the following proof that $a + b > 2$.

I. Multiplying the given equation by ab gives $a^2b + b = ab^2 + a$.

II. Rearranging and factorising gives $(a - b)(ab - 1) = 0$.

III. Since $a \neq b$, it follows that $ab = 1$.

IV. Therefore $(a + b)^2 = (a - b)^2 + 4ab > 4$.

V. Since $a + b > 0$, this gives $a + b > 2$.

Which of the following is correct?

A The first error is in line I

B The first error is in line II

C The first error is in line III

D The first error is in line IV

E The first error is in line V

F All five lines I–V are valid

11. A sequence has eight terms, each equal to 0 or 1. Exactly three terms are equal to 1, and no two of these are consecutive.

How many such sequences have at least one of the first and last terms equal to 1?

A 4

B 12

C 16

D 20

E 36

12. For a real constant k, the function f is defined for all real x by

$$f(x) = x^3 + 3x^2 + kx.$$

Consider the statements

P: $k \geq 3$.

Q: For all real numbers u and v, if $f(u) = f(v)$ then $u = v$.

Which of the following is correct?

A P is both necessary and sufficient for Q

B P is sufficient but not necessary for Q

C P is neither necessary nor sufficient for Q

D P is necessary but not sufficient for Q

13. A fair six-sided die, numbered 1 to 6, is rolled four times independently.

Given that the largest result is exactly 2 greater than the smallest result, what is the probability that the mean of the four results is halfway between the smallest and largest results?

A $\frac{3}{25}$

B $\frac{6}{25}$

C $\frac{1}{3}$

D $\frac{9}{25}$

E $\frac{18}{49}$

F $\frac{18}{25}$

14. A function f assigns a real number to each positive integer. Consider the statement:

For every positive real number a, there is a positive integer n such that $f(n) > a$ and $f(n+1) > a$.

Which of the following is the negation of this statement?

A There is a positive real number a such that, for every positive integer n, both $f(n) \leq a$ and $f(n+1) \leq a$

B For every positive real number a, there is a positive integer n such that $f(n) \leq a$ or $f(n+1) \leq a$

C There is a positive integer n such that, for every positive real number a, $f(n) \leq a$ or $f(n+1) \leq a$

D For every positive integer n, there is a positive real number a such that $f(n) \leq a$ and $f(n+1) \leq a$

E There is a positive real number a such that, for every positive integer n, $f(n) \leq a$ or $f(n+1) \leq a$

15. What is the sum of $\cos^2\theta$ over all the distinct solutions of

$$\cos^2\theta = \left|\sin\theta - \frac{1}{2}\right|$$

for which $0 \leq \theta < 2\pi$?

A $\dfrac{\sqrt{7}+\sqrt{3}-2}{2}$

B $6-\sqrt{7}-\sqrt{3}$

C $\sqrt{3}$

D $\sqrt{7}+\sqrt{3}-2$

E $\sqrt{7}-2$

F $2+\sqrt{7}-\sqrt{3}$

16. Let $f(x) = ax^2 + bx + c$, where a, b and c are real constants and $a \neq 0$.

Consider the condition

the stationary point of $y = f(x)$ lies above the x-axis.

For the equation $f(x) = 0$ to have no real solutions, this condition is

A necessary but not sufficient.

B neither necessary nor sufficient.

C sufficient but not necessary.

D both necessary and sufficient.

17. The function F is defined for all real x by

$$F(x) = \int_0^x |t^3 - 4t| \, \mathrm{d}t.$$

What is the product of the x-coordinates of all the stationary points of the graph

$$y = F(x+1) - F(x)\,?$$

A -1

B $-\dfrac{2}{3}$

C 3

D $\dfrac{3}{2}$

E $-\dfrac{3}{2}$

F $-\dfrac{1}{2}$

18. The functions f and g are defined for all real numbers, take real values, and satisfy

$$f(g(x)) = x$$

for every real x.

Consider the following statements.

I For all real u and v, if $g(u) = g(v)$, then $u = v$.

II For every real y, there is a real t such that $f(t) = y$.

III For every real x, $g(f(x)) = x$.

Which statements must be true?

A III only
B I only
C II only
D I and III only
E II and III only
F I and II only
G I, II and III
H None of them

19. The function f is defined for all real numbers. The graph

$$y = -2f(3 - x) + 5$$

has a unique minimum at $(1, -7)$.

At which point does the graph

$$y = f(2x + 1) + 4$$

have its maximum?

A $\left(-\frac{1}{2}, 10\right)$

B $\left(\frac{1}{2}, 2\right)$

C $\left(\frac{1}{2}, 10\right)$

D $(1, 10)$

E $(2, 10)$

F $(5, 10)$

20. A student begins a proof that there are no positive integers a and b satisfying

$$a^2 + b^2 = 3ab.$$

Assume that a solution exists, and choose one for which $a+b$ is as small as possible. By interchanging a and b if necessary, take $a \leq b$. In fact, $a < b$, since $a = b$ would imply $a = 0$.

The student will complete the proof by finding a number c such that

1. c is a positive integer;
2. $c < a$;
3. $c^2 + a^2 = 3ca$.

The pair c, a would then be a solution with a smaller sum.

Which expression for c completes the proof?

A $3a - b$

B $3b - a$

C $b - a$

D $2a - b$

E $\dfrac{a+b}{2}$

F $a + b$