1. C

Since $ab = 7 \neq 0$,

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{(a+b)^2 - 2ab}{ab} = \frac{36 - 14}{7} = \frac{22}{7}.$$

2. E

A counterexample must satisfy the premise and fail the conclusion. For $a = -2$ and $b = 3$, we have $-2 < 3$, but

$$\frac{1}{a} = -\frac{1}{2} < \frac{1}{3} = \frac{1}{b}.$$

Thus option E is a counterexample. In A and B the conclusion holds; in C and D the premise $a < b$ is false.

3. B

At an intersection, $\sqrt{x+4} = 2-x$. The square root requires $x \geq -4$, and its nonnegativity requires $x \leq 2$. Squaring gives

$$x + 4 = (2 - x)^2, \qquad x(x - 5) = 0.$$

The value $x = 5$ fails $x \leq 2$. The remaining value $x = 0$ satisfies the original equation and gives $y = 2$, so $P = (0, 2)$.

4. D

Write the two roots as r and s, so $f(x) = (x - r)(x - s)$. If Q holds, then $r > 1$ and $s > 1$, and hence

$$f(1) = (1 - r)(1 - s) > 0.$$

Thus P is necessary for Q. It is not sufficient: for $f(x) = x(x + 1)$, the two distinct real roots are 0 and -1, while $f(1) = 2 > 0$.

5. A

The sum of the roots of $t^2 - 6t + k$ is 6, so

$$\log_2 x + \log_2(x + 12) = 6, \qquad x(x + 12) = 2^6 = 64.$$

Therefore

$$x^2 + 12x - 64 = (x - 4)(x + 16) = 0.$$

Since $x > 0$, we have $x = 4$. The two roots in t are consequently $\log_2 4 = 2$ and $\log_2 16 = 4$. Their product is $k = 8$.

6. C

An implication is logically equivalent to its contrapositive. The negation of $f(x) > 1$ is $f(x) \leq 1$. The negation of the condition $x < 0$ or $x > 3$ is the condition $x \geq 0$ and $x \leq 3$. Thus the equivalent statement is: for every real x, if $f(x) \leq 1$, then $0 \leq x \leq 3$.

7. D

Let the rectangle have width $w = GF$ and height h. Since GF is parallel to BC, triangles AGF and ABC are similar. Their heights are $8 - h$ and 8, so

$$\frac{w}{12} = \frac{8-h}{8}, \qquad w = 12 - \frac{3}{2}h.$$

The perimeter gives $w + h = 10$. Hence

$$12 - \frac{3}{2}h + h = 10, \qquad h = 4, \qquad w = 6.$$

The required area is $wh = 24$.

8. D

Suppose P holds. Since $(a-b)^2 \geq 0$,

$$(a+1)(b+1) = \frac{(a+b+2)^2 - (a-b)^2}{4} \leq \frac{(a+b+2)^2}{4} \leq 9.$$

Thus P is sufficient for Q. It is not necessary: $a = 5$ and $b = \frac{1}{2}$ give $(a+1)(b+1) = 9$, but $a + b = \frac{11}{2} > 4$. Hence the answer is D.

9. C

The conditions give $0 \leq x \leq 2$. For each such x, the upper boundary is the lower of $y = x^2$ and $y = 2 - x$.

These graphs meet when

$$x^2 = 2 - x, \qquad (x-1)(x+2) = 0.$$

Only $x = 1$ is in the region. Since $x^2 - (2 - x) = (x - 1)(x + 2)$, the parabola is the lower graph for $0 \leq x \leq 1$, and the line is the lower graph for $1 \leq x \leq 2$.

The required area is therefore

$$\int_0^1 x^2 \, \mathrm{d}x + \int_1^2 (2 - x) \, \mathrm{d}x = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The answer is C.

10. F

Every line is valid. Positivity makes the reciprocals well-defined, and multiplying by ab gives line I. Expanding $(a-b)(ab-1)$ verifies line II. Distinctness gives $a-b \neq 0$, so line III follows. It also gives $(a-b)^2 > 0$, which together with $ab = 1$ justifies the strict inequality in line IV. Finally, $a+b > 0$ selects the positive possibility in line V. Thus the answer is F.

11. C

If the positions of the three 1s are $i < j < k$, nonconsecutiveness means that

$$i, \quad j-1, \quad k-2$$

are any three distinct increasing integers from 1 to 6. There are $\binom{6}{3} = 20$ choices.

If both end terms are 0, the three 1s must occupy nonconsecutive positions among the six interior terms. The same argument gives $\binom{4}{3} = 4$ choices. Hence the required number is

$$20 - 4 = 16,$$

so the answer is C.

12. A

Writing $x = t - 1$ gives

$$f(t-1) = t^3 + (k-3)t + 2 - k.$$

First suppose $k \geq 3$. For distinct real s and t,

$$f(s-1) - f(t-1) = (s-t)\big(s^2 + st + t^2 + k - 3\big).$$

The bracket is positive, since

$$s^2 + st + t^2 = \frac{(s+t)^2 + s^2 + t^2}{2} > 0$$

when $s \neq t$, and $k - 3 \geq 0$. Thus distinct inputs have distinct outputs, proving sufficiency.

If $k < 3$, put $r = \sqrt{3-k} > 0$. Then

$$f(-1) = 2 - k = f(r-1),$$

since $r^3 + (k-3)r = 0$. These are distinct inputs, so Q fails. This proves necessity, including the boundary $k = 3$ in the sufficient case. The answer is A.

13. D

Fix the smallest result as m. The largest is then $m+2$, where m is one of $1, 2, 3, 4$. All four results belong to $m, m+1, m+2$, and both extreme values must occur. There are

$$3^4 - 2 \cdot 2^4 + 1 = 50$$

ordered outcomes: subtract outcomes missing either extreme, then restore the all-$(m+1)$ outcome subtracted twice.

The required mean is $m+1$. Thus the number of results equal to m must equal the number equal to $m+2$. There are exactly two possibilities:

- One of each extreme and two middle results: $4 \cdot 3 = 12$ orders.
- Two of each extreme: $\binom{4}{2} = 6$ orders.

Every ordered four-roll outcome is equally likely. Each of the four values of m gives the same counts, so the conditional probability is

$$\frac{4(12+6)}{4 \cdot 50} = \frac{9}{25}.$$

The answer is D.

14. E

The statement fails precisely when there is a positive threshold a for which no positive integer n makes both inequalities true. For that threshold, every positive integer n must therefore satisfy

$$f(n) \leq a \quad \text{or} \quad f(n+1) \leq a.$$

The same a is used for all n, while which of the two inequalities holds may depend on n. This is option E.

15. D

Put $s = \sin\theta$, so $-1 \le s \le 1$. The equation becomes

$$1 - s^2 = \left| s - \frac{1}{2} \right|.$$

If $s \ge \frac{1}{2}$, then $s^2 + s - \frac{3}{2} = 0$. Of its roots, only

$$s = \frac{-1 + \sqrt{7}}{2}$$

lies in the required range. If $s < \frac{1}{2}$, then $s^2 - s - \frac{1}{2} = 0$, and only

$$s = \frac{1 - \sqrt{3}}{2}$$

is admissible. Both retained values lie strictly between -1 and 1, and each gives two distinct angles with $0 \le \theta < 2\pi$.

For the first value, $2(1 - s^2) = \sqrt{7} - 2$; for the second, $2(1 - s^2) = \sqrt{3}$. Hence the required sum is

$$\boxed{\sqrt{7} + \sqrt{3} - 2}.$$

16. B

For $f(x) = 1 - x^2$, the stationary point is $(0, 1)$, but $f(x) = 0$ has the real solutions $x = \pm 1$. Thus the condition is not sufficient.

For $f(x) = -1 - x^2$, there are no real zeros, but the stationary point is $(0, -1)$. Thus the condition is not necessary.

It is therefore neither necessary nor sufficient.

17. E

The Fundamental Theorem of Calculus gives $F'(x) = |x^3 - 4x|$. The graph of $F(x+1)$ is a horizontal translation of the graph of $F(x)$, so its gradient at x is $F'(x+1)$. Hence a stationary point requires

$$|(x+1)^3 - 4(x+1)| = |x^3 - 4x|.$$

Two real numbers have equal absolute values exactly when they are equal or opposite. The equal case gives

$$3(x^2 + x - 1) = 0,$$

while the opposite case gives

$$2x^3 + 3x^2 - 5x - 3 = (2x+1)(x^2 + x - 3) = 0.$$

The two quadratics have positive discriminants 5 and 13, respectively. They have no common root, and $-\frac{1}{2}$ is a root of neither, so these equations give five distinct real stationary abscissae.

By comparing constant terms in their factorizations, the products of the roots of $x^2 + x - 1$ and $x^2 + x - 3$ are -1 and -3. The required product is therefore

$$(-1)\left(-\frac{1}{2}\right)(-3) = -\frac{3}{2}.$$

18. F

If $g(u) = g(v)$, applying f gives $f(g(u)) = f(g(v))$, and hence $u = v$. Thus statement I is true.

For any real y, taking $t = g(y)$ gives $f(t) = y$. Thus statement II is true.

For a counterexample to statement III, define

$$f(x) = \begin{cases} x + 1, & x < 0, \\ x, & x \geq 0, \end{cases} \qquad g(x) = \begin{cases} x - 1, & x < 0, \\ x, & x \geq 0. \end{cases}$$

If $x < 0$, then $f(g(x)) = f(x - 1) = x$; if $x \geq 0$, then $f(g(x)) = f(x) = x$. Thus the hypothesis holds for every real x. However,

$$g\left(f\left(-\frac{1}{2}\right)\right) = g\left(\frac{1}{2}\right) = \frac{1}{2} \neq -\frac{1}{2}.$$

Only statements I and II must be true.

19. C

At $x = 1$, the input to f is $3 - 1 = 2$, and

$$-2f(2) + 5 = -7,$$

so $f(2) = 6$. Since multiplication by -2 reverses inequalities and $3 - x$ takes every real value, f has its unique maximum value 6 at input 2.

The graph $y = f(2x + 1) + 4$ therefore has its maximum when

$$2x + 1 = 2, \qquad x = \frac{1}{2},$$

and the maximum value is $6 + 4 = 10$. The point is

$$\boxed{\left(\frac{1}{2}, 10\right)}.$$

20. A

Take $c = 3a - b$, which is an integer. The original equation gives

$$b(3a - b) = a^2,$$

so $c = a^2/b > 0$. Since $a < b$, it also gives $c < a$.

Moreover,

$$\begin{aligned} c^2 + a^2 - 3ca &= (3a - b)^2 + a^2 - 3a(3a - b) \\ &= a^2 + b^2 - 3ab = 0. \end{aligned}$$

Thus c, a are positive integers satisfying the same equation, while

$$c + a < a + b.$$

This contradicts the minimal choice of $a + b$, completing the proof.