1. A

Let r be the common real solution. Subtracting the first equation from the second at $x = r$ gives

$$3r - 9 = 0,$$

so $r = 3$. Substitution into the first equation gives

$$9 + 3a + 12 = 0,$$

hence $a = -7$. This value does give a common solution: the equations become $(x - 3)(x - 4) = 0$ and $(x - 1)(x - 3) = 0$.

2. B

The three means give

$$a+b+c+d+e+f=60, \qquad a+b+c+d=28, \qquad c+d+e+f=48.$$

Adding the last two equations counts c and d twice and each other number once. Therefore

$$c+d=28+48-60=16.$$

The numbers are already in order, so their median is $(c+d)/2=8$.

3. E

The centre lies on the perpendicular bisector $x = 4$ of OA. The radius to O is perpendicular to the tangent $y = -2x$, so the centre also lies on $y = x/2$. Thus the centre is $(4, 2)$, and the circle has equation

$$(x-4)^2 + (y-2)^2 = 20.$$

Putting $y = x$ gives

$$2x^2 - 12x = 0.$$

The intersections are O and $P = (6, 6)$. Triangle OAP has base $OA = 8$ and perpendicular height 6, so its area is $\frac{1}{2} \times 8 \times 6 = 24$.

4. B

Differentiating gives

$$\frac{dy}{dx} = 3x^2 + 2ax.$$

At $x = 2$, the gradient condition is $12 + 4a = 4$, so $a = -2$. The point of tangency is then $(2, 8 - 8) = (2, 0)$. The tangent has equation

$$y = 4(x - 2) = 4x - 8,$$

so its y-intercept is -8.

5. C

Let $S_n = 2n^2 + 3n$. For $n \geq 2$,

$$u_n = S_n - S_{n-1} = 2n^2 + 3n - \big(2(n-1)^2 + 3(n-1)\big) = 4n + 1.$$

This formula also holds for $n = 1$, since $u_1 = S_1 = 5$. Thus $u_n > 101$ is equivalent to $n > 25$. The least possible integer is 26; indeed $u_{25} = 101$ and $u_{26} = 105$.

6. D

Factor the equation as

$$\cos\theta\,(2\cos\theta - 3\sin\theta) = 0.$$

The first factor vanishes at $\pi/2$ and $3\pi/2$, giving two contributions of 1 to the required sum. In the other case, $2\cos\theta = 3\sin\theta$, so

$$4\cos^2\theta = 9\sin^2\theta, \qquad 4(1-\sin^2\theta) = 9\sin^2\theta.$$

Hence $\sin^2\theta = 4/13$. There are two such angles, since $\tan\theta = 2/3$ has one solution in each of the first and third quadrants. Neither belongs to the first case. The sum is

$$1 + 1 + \frac{4}{13} + \frac{4}{13} = \frac{34}{13}.$$

7. F

Replacing x by $1-x$ in the given identity gives

$$f(1-x)+2f(x)=(1-x)^2.$$

Twice this equation minus the original equation yields

$$3f(x)=2(1-x)^2-x^2=x^2-4x+2.$$

Consequently $f(|x|)=k$ is equivalent to

$$(|x|-2)^2=3k+2.$$

For four distinct real solutions, there must be two distinct positive values of $|x|$, each giving a pair of opposite nonzero values of x. These values are

$$2+\sqrt{3k+2} \quad \text{and} \quad 2-\sqrt{3k+2}.$$

They are distinct and both positive exactly when

$$0<\sqrt{3k+2}<2,$$

which is equivalent to $-2/3<k<2/3$. At the lower endpoint there are only two solutions, and at the upper endpoint $|x|=0$ or 4, giving only three. Thus neither endpoint is included.

8. D

Let E be the event that the score is at least 6, and let T be the event that there are two rolls.

For T and E both to occur, the first result can be 1, 2 or 3. These require the second result to be at least 5, 4 or 3, respectively. Thus

$$\Pr(T \text{ and } E) = \frac{1}{6}\frac{2}{6} + \frac{1}{6}\frac{3}{6} + \frac{1}{6}\frac{4}{6} = \frac{1}{4}.$$

If there is only one roll, E occurs only when that roll is 6, which has probability 1/6. Hence

$$\Pr(E) = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}, \qquad \Pr(T \mid E) = \frac{1/4}{5/12} = \boxed{\frac{3}{5}}.$$

9. E

Let M and P be the midpoints of AB and CD, respectively. A perpendicular from the centre to a chord bisects the chord, so

$$AM = 3, \qquad CP = 4.$$

Pythagoras' theorem gives

$$OM = \sqrt{5^2 - 3^2} = 4, \qquad OP = \sqrt{5^2 - 4^2} = 3.$$

Since the parallel chords are on opposite sides of O, their perpendicular separation is

$$OM + OP = \boxed{7}.$$

10. C

At $x = 1$, the tangent has gradient $6(1) - 4 = 2$, so the normal has gradient $-1/2$. As it passes through $(0, 5)$, the normal has equation

$$y = 5 - \frac{x}{2}.$$

Its point of contact with the curve has $x = 1$, so $f(1) = 9/2$.

Integrating the given derivative gives

$$f(x) = 3x^2 - 4x + C.$$

Therefore $-1 + C = 9/2$, so $C = 11/2$. It follows that

$$\int_0^2 f(x)\,\mathrm{d}x = \left[x^3 - 2x^2 + \frac{11x}{2}\right]_0^2 = 8 - 8 + 11 = \boxed{11}.$$

11. A

Squaring $x + y + z = 0$ gives

$$0 = 14 + 2(xy + yz + zx), \qquad xy + yz + zx = -7.$$

Also, $z = -x - y$, so direct expansion gives

$$x^3 + y^3 + z^3 = x^3 + y^3 - (x + y)^3 = -3xy(x + y) = 3xyz.$$

Thus $3xyz = -18$, and $xyz = -6$. Hence

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{xy + yz + zx}{xyz} = \frac{-7}{-6} = \boxed{\frac{7}{6}}.$$

12. E

Pair four factors of $1 - x$ with four factors of $1 + x$:

$$(1 - x)^4(1 + x)^7 = (1 - x^2)^4(1 + x)^3.$$

In $(1-x^2)^4$, only the terms in x^2 and x^4 can contribute to x^5 after multiplication by $(1+x)^3$. Their coefficients are -4 and 6.

In $(1 + x)^3$, the coefficients of x^3 and x are 1 and 3. Therefore the required coefficient is

$$(-4)(1) + (6)(3) = \boxed{14}.$$

13. C

The centre is $(18+2)/2 = 10$ metres above the ground, and the radius is $(18-2)/2 = 8$ metres.

At either point where the height is 14 metres, the radius makes an acute angle θ with the upward vertical. Hence

$$\cos\theta = \frac{14-10}{8} = \frac{1}{2}, \qquad \theta = 60^\circ.$$

The arc above this height therefore subtends $2\theta = 120^\circ$ at the centre. Since the speed is constant, the time spent on this arc is

$$36 \times \frac{120}{360} = \boxed{12} \text{ seconds.}$$

Excluding the two instants at height 14 metres does not change the duration.

14. B

Let a and b be the exponents of 2 and 3 in the prime factorisation of N.

Since $N/12$ is an integer square, $a-2$ and $b-1$ must be nonnegative even integers. In particular, a is even and b is odd.

Since $N/18$ is an integer cube, $a-1$ and $b-2$ must be nonnegative multiples of 3. The possible exponents of 2 from this condition are $1, 4, 7, \ldots$, of which the smallest even one is 4. The possible exponents of 3 are $2, 5, 8, \ldots$, of which the smallest odd one is 5.

Thus every admissible N has at least the factor $2^4 3^5$. Any other prime factor would only increase N. The value $N = 2^4 3^5$ works, since

$$\frac{N}{12} = 2^2 3^4 = 18^2, \qquad \frac{N}{18} = 2^3 3^3 = 6^3.$$

Therefore the least value is $\boxed{2^4 3^5}$, which is 3888.

15. A

The graph gives

$$f(u) > 0 \quad \Longleftrightarrow \quad -1 < u < 0 \text{ or } 3 < u < 8.$$

Putting $u = x^2 - 1$, these conditions become

$$0 < x^2 < 1 \quad \text{or} \quad 4 < x^2 < 9.$$

Thus the required values satisfy one of

$$-3 < x < -2, \qquad -1 < x < 0, \qquad 0 < x < 1, \qquad 2 < x < 3.$$

Each part has length 1, so the total length is $\boxed{4}$.

All these values put $x^2 - 1$ inside the domain of f.

16. E

A circle of radius $r > 0$ tangent to both axes, with centre in the first quadrant, has centre (r, r). The distance d between its centre and the centre $(3, 3)$ of C satisfies

$$d^2 = 2(r-3)^2.$$

For external tangency, $d = r + 1$. Hence

$$2(r-3)^2 = (r+1)^2 \quad \Longleftrightarrow \quad r^2 - 14r + 17 = 0.$$

Its roots are $7 - 4\sqrt{2}$ and $7 + 4\sqrt{2}$. Both are positive, so both give valid external tangencies. Their sum is 14.

For internal tangency, $d = |r - 1|$. Hence

$$2(r-3)^2 = (r-1)^2 \quad \Longleftrightarrow \quad r^2 - 10r + 17 = 0.$$

Its roots are $5 - 2\sqrt{2}$ and $5 + 2\sqrt{2}$, both greater than 1. Both therefore give valid circles containing C and tangent to it internally. Their sum is 10.

The two cases have no common positive radius: subtracting their equations would give $4r = 0$. These are all possible tangencies, so the required sum is

$$\boxed{14 + 10 = 24}.$$

17. D

There are two mutually exclusive ways for the colours to differ.

For red followed by blue, the red counter is replaced, so the probability is

$$\frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$$

For blue followed by red, there are then three red counters among four counters, so the probability is

$$\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}.$$

Adding gives

$$\frac{6}{25} + \frac{3}{10} = \boxed{\frac{27}{50}}.$$

18. B

The coordinate is positive for $0 < t < 2$ and negative for $2 < t < 5$. Moving towards the origin therefore requires a negative velocity in the first part and a positive velocity in the second.

Expanding before differentiating gives

$$x(t) = t^3 - 7t^2 + 10t, \qquad x'(t) = 3t^2 - 14t + 10.$$

The two zeros of the velocity are

$$\alpha = \frac{7 - \sqrt{19}}{3}, \qquad \beta = \frac{7 + \sqrt{19}}{3},$$

with $0 < \alpha < 2 < \beta < 5$. Since the quadratic has positive leading coefficient, the velocity is negative for $\alpha < t < \beta$ and positive outside these limits.

Consequently the particle moves towards the origin for

$$\alpha < t < 2 \qquad \text{and} \qquad \beta < t < 5.$$

The total time is

$$(2 - \alpha) + (5 - \beta) = 7 - (\alpha + \beta) = 7 - \frac{14}{3} = \boxed{\frac{7}{3}} \text{ seconds.}$$

The isolated times at the origin or with zero velocity do not affect this duration.

19. C

Taking reciprocals in the recurrence gives

$$\frac{1}{u_{k+1}} = \frac{1+u_k}{u_k} = \frac{1}{u_k} + 1.$$

Thus the reciprocals form an arithmetic progression with first term 2 and common difference 1, so $1/u_k = k+1$.

The product condition gives

$$(n+1)(n+2) = 156,$$

or

$$n^2 + 3n - 154 = (n-11)(n+14) = 0.$$

Since n is positive, $n = 11$. The required sum is therefore

$$2 + 3 + \cdots + 12 = \frac{11(2+12)}{2} = \boxed{77}.$$

20. F

The quadratic being squared has leading coefficient 1 or -1. Negating it if necessary, write it as $x^2 + px + q$. Expanding gives

$$(x^2 + px + q)^2 = x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2.$$

Comparison of coefficients yields

$$a = 2p, \qquad p^2 + 2q = 2p + 3, \qquad b = 2pq, \qquad q^2 = 4.$$

There are two cases.

If $q = 2$, then $p^2 - 2p + 1 = 0$, so $p = 1$ and $b = 4$.

If $q = -2$, then $p^2 - 2p - 7 = 0$, so $p = 1 \pm 2\sqrt{2}$. Therefore

$$b = -4p = -4 \mp 8\sqrt{2}.$$

All three resulting values of b are distinct, and substitution in the square above verifies that each is possible. Their sum is

$$4 + (-4 + 8\sqrt{2}) + (-4 - 8\sqrt{2}) = \boxed{-4}.$$