1. Let a and b be real constants, and define

$$q(x) = x^2 + ax + b.$$

Consider the following statement.

For every real number x, at least one of $q(x)$ and $q(2x)$ is strictly positive.

Which of the following correctly describes the condition $a^2 < 4b$ in relation to this statement?

A The condition is necessary but not sufficient for the statement.

B The condition is sufficient but not necessary for the statement.

C The condition is both necessary and sufficient for the statement.

D The condition is neither necessary nor sufficient for the statement.

2. For a real constant k, the equation

$$2^{-|x|} + 3\,2^{-|x-2|} = k$$

has four distinct real solutions x_1, x_2, x_3, x_4.

What is the value of

$$2^{x_1+x_2+x_3+x_4}\,?$$

A $\frac{4}{3}$

B $\frac{208}{21}$

C $\frac{52}{7}$

D 16

E $\frac{208}{7}$

F $\frac{208}{3}$

3. Each use of either of two machines produces one of the numbers 0, 1 and 2, each with positive probability. The fixed probabilities may be different for the two machines. Let X and Y be the results of independent uses of the first and second machines.

The two probability-weighted means are equal:

$$\Pr(X = 1) + 2\Pr(X = 2) = \Pr(Y = 1) + 2\Pr(Y = 2) = m.$$

It is also given that

$$\Pr(X > Y) = \Pr(Y > X).$$

Consider the following statements.

I. If $\Pr(X = 0) \neq \Pr(Y = 0)$, then $m = 1$.

II. $\Pr(X = Y) \geq \dfrac{1}{3}$.

III. If $\Pr(X = 0) = \Pr(X = 2)$, then $\Pr(Y = 0) = \Pr(Y = 2)$.

Which of these statements must be true?

A I only

B II only

C III only

D I and II only

E I and III only

F I, II and III

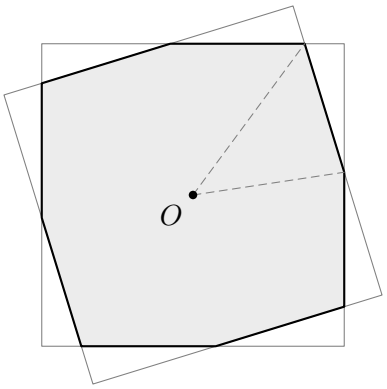
4. Two squares of side length 1 have the same centre O. One is obtained by rotating the other through an angle θ, where

$$0 < \theta < \frac{\pi}{4}.$$

Their common region is an octagon of area $8/9$. The distances from O to its eight vertices are $d_1, d_2, \ldots, d_8$.

What is the value of

$$d_1^2 + d_2^2 + \cdots + d_8^2\,?$$



A $\dfrac{104}{81}$

B $\dfrac{190}{81}$

C $\dfrac{208}{81}$

D $\dfrac{217}{81}$

E $\dfrac{226}{81}$

F $\dfrac{244}{81}$

5. Let m be a real constant. Consider the following statement.

For every list of five distinct real numbers with mean m and median 1, the median of their five squares is strictly greater than 1.

Which of the following correctly describes the condition $m < 1$ in relation to this statement?

A The condition is necessary but not sufficient for the statement.

B The condition is sufficient but not necessary for the statement.

C The condition is both necessary and sufficient for the statement.

D The condition is neither necessary nor sufficient for the statement.

6. Initially, tank A contains 1 litre of pure dye, and tank B contains 2 litres of water. A cycle consists of transferring t litres from A to B, mixing thoroughly, then transferring t litres from B back to A and mixing thoroughly again. The same value of t is used in each cycle, with $0 < t < 1$.

Let c_n be the proportion by volume of dye in tank A after n complete cycles. It is given that

$$\frac{c_1}{c_3} = \frac{8}{5}.$$

What is the sum of all the possible values of t?

A $\frac{2}{5}$

B $2 - \frac{4\sqrt{7}}{7}$

C $\frac{4}{5}$

D $\frac{1+\sqrt{7}}{4}$

E $\frac{34}{3} - 4\sqrt{7}$

F $\frac{12}{5} - \frac{4\sqrt{7}}{7}$

7. A real-valued function f, defined for every real number, satisfies

$$f(x) + f(-x) = 2x^2 + 2, \qquad f(x)f(-x) = x^4 + x^2 + 1$$

for every real x.

Consider the following statements.

I. f must be a quadratic polynomial.

II. The least value of f is $3/4$.

III. For every real $y > 3/4$, the equation $f(x) = y$ has exactly two distinct real solutions.

Which of these statements must be true?

A I only
B II and III only
C I and II only
D I and III only
E II only
F I, II and III

8. Triangle ABC has

$$AB = \sqrt{10}, \qquad BC = \sqrt{17}, \qquad CA = \sqrt{5}.$$

A point P lies strictly inside the triangle and satisfies

$$\angle APB = 90^\circ, \qquad \angle BPC = \angle CPA = 135^\circ.$$

What is the value of $PA + PB + PC$?

A 4

B $4 + \sqrt{2}$

C $3 + \sqrt{2}$

D $4 + 2\sqrt{2}$

E 6

F $\sqrt{22}$

9. Let p and q be relatively prime positive integers, and let b, c, d be rational numbers. Define

$$P(x) = \frac{p}{q}x^3 + bx^2 + cx + d.$$

Consider the following statement.

There are infinitely many positive integers n for which

$$\frac{P(n)}{n^2+1}$$

is an integer.

Which of the following correctly describes the condition that $c = p/q$, $d = b$, and pb is an integer, in relation to this statement?

A The condition is necessary but not sufficient for the statement.

B The condition is sufficient but not necessary for the statement.

C The condition is both necessary and sufficient for the statement.

D The condition is neither necessary nor sufficient for the statement.

10. A point P is chosen at random inside an equilateral triangle ABC of side length 2. The probability that P lies in a region is proportional to that region's area.

Let d_1, d_2, d_3 be the perpendicular distances from P to the lines AB, BC and CA, respectively.

Given that $d_1 \geq \sqrt{3}/2$, what is the probability that

$$d_1^2 + d_2^2 + d_3^2 \leq \frac{3}{2}\,?$$

A $\dfrac{\pi}{9\sqrt{3}} - \dfrac{1}{12}$

B $\dfrac{2\pi}{9\sqrt{3}} - \dfrac{1}{6}$

C $\dfrac{4\pi}{9\sqrt{3}} - \dfrac{1}{3}$

D $\dfrac{\pi}{3\sqrt{3}}$

E $\dfrac{4\pi}{9\sqrt{3}}$

F $\dfrac{4\pi}{9\sqrt{3}} - \dfrac{1}{6}$

11. Three events A, B and C each have probability $2/3$. Each pair of these events is independent.

A student writes the following argument, where u is the probability that at least one of the three events occurs.

I. The probability that A does not occur is $1/3$, and likewise for B and C.

II. For any two of the events, the probability that neither occurs is $1/9$.

III. Therefore the probability that none of the three events occurs is $1/27$.

IV. Hence $u = 26/27$.

Which option identifies the first invalid deduction and gives the complete set of possible values of u?

A There is no invalid deduction; $u = 26/27$.

B Line II; $\dfrac{2}{3} \leq u \leq 1$.

C Line III; $\dfrac{8}{9} \leq u \leq 1$.

D Line III; $\dfrac{2}{3} \leq u \leq 1$.

E Line III; $\dfrac{8}{9} \leq u \leq \dfrac{26}{27}$.

F Line IV; $u = 8/9$.

12. A quadratic polynomial f with real coefficients satisfies

$$\int_0^1 f(x)^2 \, \mathrm{d}x = 1, \qquad \int_0^1 f(x) f(1-x) \, \mathrm{d}x = 0.$$

What is the greatest possible value of $f(1/2)$?

A $\dfrac{\sqrt{2}}{2}$

B 1

C $\dfrac{\sqrt{3}}{2}$

D $\dfrac{\sqrt{6}}{2}$

E $\dfrac{3}{2}$

F $\dfrac{3\sqrt{2}}{4}$

13. For $0 < x < 1$ and each positive integer n, define

$$F_n(x) = nx^n(1 - x).$$

Consider the following statements.

I. For every real x with $0 < x < 1$, there exists a positive integer N such that $F_n(x) < 1/4$ for every integer $n \geq N$.

II. There exists a positive integer N such that $F_n(x) < 1/4$ for every real x with $0 < x < 1$ and every integer $n \geq N$.

III. For every positive integer n, there exists a real x with $0 < x < 1$ such that $F_n(x) \geq 1/4$.

Which of these statements are true?

A I only

B III only

C I and II only

D II and III only

E I and III only

F I, II and III

14. Let $\theta_1, \ldots, \theta_m$ be all the distinct real solutions with $0 \leq \theta < 2\pi$ of

$$\sin\left(\frac{\pi}{2}(2\sin\theta + \cos\theta)\right) = \cos\left(\frac{\pi}{2}(\sin\theta - 2\cos\theta)\right).$$

What is the value of

$$\sin\theta_1 + \cdots + \sin\theta_m\,?$$

A $-\frac{4}{5}$

B 0

C $-\frac{2}{5}$

D $-\frac{6}{5}$

E $-\frac{8}{5}$

F $\frac{4}{5}$

15. The distinct cubic polynomials P and Q both have coefficient 1 for x^3. Neither polynomial has a repeated real root, and

$$P(x)Q(x) \geq 0 \qquad \text{for every real } x.$$

The curves $y = P(x)$ and $y = Q(x)$ intersect at exactly two distinct points.

Consider the following statements.

I. Exactly one of the two intersections lies on the x-axis.

II. The straight line joining the two intersections has positive gradient.

III. At each of the two intersections, the tangents to both curves have positive gradient.

Which of these statements must be true?

A I only

B II only

C I and III only

D II and III only

E I and II only

F I, II and III

16. Four coins A, B, C, D are tossed independently. Each coin has a fixed probability of heads strictly between 0 and 1, and coin A has probability $1/3$ of heads.

The probability of no heads equals the probability of four heads. The probability of exactly one head equals the probability of exactly three heads.

What is the greatest possible probability that coin A shows heads, given that exactly two of the four coins show heads?

A $\frac{1}{9}$

B $\frac{2}{7}$

C $\frac{1}{5}$

D $\frac{1}{3}$

E $\frac{7}{18}$

F $\frac{1}{2}$

17. The positive real numbers a, b, c, d satisfy

$$ab = 1, \qquad bc = 2, \qquad cd = 6, \qquad da = 3.$$

Consider the two conditions

S: $a + b + c + d$ is rational.

T: $ac + bd$ is rational.

Which of the following correctly describes when these conditions are sufficient to guarantee that all four numbers a, b, c, d are rational?

A S alone is sufficient, but T alone is not.

B T alone is sufficient, but S alone is not.

C Each of S and T alone is sufficient.

D S and T together are sufficient, but neither alone is sufficient.

E Even S and T together are not sufficient.

18. A particle moves in a plane. At time $t \geq 0$, its coordinates are $x(t)$ and $y(t)$, where

$$x(0) = y(0) = 0, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 1 - t^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2t.$$

Its speed is

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}.$$

For $T > 0$, let $D(T)$ be the total distance travelled during $0 \leq t \leq T$, and let $R(T)$ be the straight-line distance of the particle from the origin at time T.

What is the greatest possible value of $D(T)/R(T)$?

A $\dfrac{2}{\sqrt{3}}$

B 1

C $\dfrac{4}{\sqrt{13}}$

D $\dfrac{4}{3}$

E $\sqrt{2}$

19. Let n be a positive integer. Consider the following statement.

There exists a polynomial P with real coefficients and degree exactly 3 such that, for every integer k with $1 \leq k \leq n$,

$$P(k) = k \qquad \text{or} \qquad P(k) = k^2.$$

The choice between the two equalities may be different for different values of k.

Which of the following correctly describes the condition $n \leq 4$ in relation to this statement?

A The condition is necessary but not sufficient for the statement.

B The condition is sufficient but not necessary for the statement.

C The condition is both necessary and sufficient for the statement.

D The condition is neither necessary nor sufficient for the statement.

20. The vertices of a rectangle are A, B, C, D, in that order, with

$$AB = 2\sqrt{3}, \qquad BC = 2.$$

A point P in the plane of the rectangle satisfies

$$PA^2 + PC^2 = 12.$$

What is the least possible value of

$$PA \cdot PB \cdot PC \cdot PD\,?$$

A 4

B 12

C $6\sqrt{3}$

D $4\sqrt{13}$

E 28

F 108