**1. B**

If $a^2 < 4b$, then completing the square gives

$$q(x) = \left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4} > 0$$

for every real $x$. In particular, at least one of $q(x)$ and $q(2x)$ is strictly positive for every real $x$. Thus the condition is sufficient.

It is not necessary. Take $a = 3$ and $b = 9/4$. Then

$$q(x) = \left(x + \frac{3}{2}\right)^2, \qquad q(2x) = \left(2x + \frac{3}{2}\right)^2.$$

Both values are non-negative. The first can be zero only at $x = -3/2$, and the second only at $x = -3/4$, so they cannot both be zero at the same $x$. The statement is therefore true, but $a^2 = 4b$, so the proposed strict inequality is false.

The condition is sufficient but not necessary, giving **B**.

**2. B**

Put $t = 2^x$, so that $t > 0$ and distinct values of $x$ correspond to distinct values of $t$. The three parts of the equation become

| condition | equation |
|---|---|
| $0 < t \leq 1$ | $\dfrac{7t}{4} = k$ |
| $1 < t < 4$ | $\dfrac{1}{t} + \dfrac{3t}{4} = k$ |
| $t \geq 4$ | $\dfrac{13}{t} = k.$ |

Each outer part has at most one solution, and the middle part has at most two. Four distinct solutions therefore require one from each outer part and two from the middle part. Their $t$-values on the outer parts are

$$\frac{4k}{7}, \qquad \frac{13}{k},$$

where $k > 0$. The two middle values are roots of

$$3t^2 - 4kt + 4 = 0,$$

so their product is $4/3$. It follows that

$$2^{x_1+x_2+x_3+x_4} = \frac{4k}{7} \cdot \frac{4}{3} \cdot \frac{13}{k} = \boxed{\frac{208}{21}}.$$

The premise is attainable: for example, $k = 26/15$ gives the four distinct $t$-values $104/105$, $10/9$, $6/5$ and $15/2$, all in their required parts.

**3. E**

Write $c = \Pr(X = 2)$ and $f = \Pr(Y = 2)$. Using the common mean and the fact that each set of probabilities sums to 1, the probabilities for outputs $0, 1, 2$ are

$$(1 - m + c,\ m - 2c,\ c), \qquad (1 - m + f,\ m - 2f,\ f).$$

Independence gives

$$\Pr(X > Y) = (m - 2c)(1 - m + f) + c(1 - f),$$
$$\Pr(Y > X) = (m - 2f)(1 - m + c) + f(1 - c).$$

Subtracting and simplifying gives

$$0 = (m - 1)(c - f).$$

Different probabilities for output 0 mean $c \neq f$, and therefore $m = 1$. Statement I is true.

For statement II, take the two probability triples to be

$$\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \qquad \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right).$$

All probabilities are positive. Both means are 1, and the identity above shows that the two winning probabilities are equal. However,

$$\Pr(X = Y) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10} < \frac{1}{3}.$$

Statement II is false.

Finally, $\Pr(X = 0) = \Pr(X = 2)$ means $1 - m + c = c$, so $m = 1$. The second probability triple then also has equal first and third entries. Statement III is true. Thus exactly I and III must hold.

**4. C**

Place $O$ at the origin and the sides of the first square on $x = \pm 1/2$ and $y = \pm 1/2$. Write $c = \cos\theta$ and $s = \sin\theta$, and put

$$u = \frac{1-c}{s}, \qquad v = \frac{1-s}{c}.$$

Here $0 < s < c < 1$. The rotated side with equation $cx + sy = 1/2$ meets the right and top sides of the first square at

$$\left(\frac{1}{2}, \frac{u}{2}\right), \qquad \left(\frac{v}{2}, \frac{1}{2}\right).$$

These points and their rotations through multiples of $\pi/2$ are the eight vertices of the common region. In particular, four vertices have squared distance $(1+u^2)/4$ and four have squared distance $(1+v^2)/4$.

Using $c^2 + s^2 = 1$,

$$u + v + uv = \frac{c(1-c) + s(1-s) + (1-c)(1-s)}{sc} = 1.$$

The four corners removed from the first square are congruent right-angled triangles. Each has legs $(1-u)/2$ and $(1-v)/2$. Thus the area of the common region is

$$1 - \frac{(1-u)(1-v)}{2} = 1 - uv = u + v.$$

Consequently $u + v = 8/9$ and $uv = 1/9$. The required sum is therefore

$$4\frac{1+u^2}{4} + 4\frac{1+v^2}{4} = 2 + (u+v)^2 - 2uv = 2 + \frac{64}{81} - \frac{2}{9} = \boxed{\frac{208}{81}}.$$

The given area is geometrically attainable. Taking $c = (5+\sqrt{7})/8$ and $s = (5-\sqrt{7})/8$ gives $c^2 + s^2 = 1$ and $0 < s < c$, and yields $u + v = 8/9$.

**5. A**

Write the original numbers in increasing order as

$$a < b < 1 < d < e.$$

Suppose that the median of the squares is at most 1. Since $d^2 > 1$ and $e^2 > 1$, the other three squares must all be at most 1. In particular, $a \geq -1$. The strict inequalities between the original numbers now give

$$a + b > -2, \qquad d + e > 2.$$

Hence

$$5m = a + b + 1 + d + e > 1.$$

By the contrapositive, if $m \leq 1/5$, the median of the squares must be greater than 1.

To show that no larger $m$ works, suppose $m > 1/5$. Choose $\varepsilon$ with

$$0 < \varepsilon < \frac{1}{2}, \qquad 5\varepsilon < 5m - 1.$$

Consider the five numbers

$$-1 + \varepsilon, \quad -1 + 2\varepsilon, \quad 1, \quad 1 + \varepsilon, \quad 5m - 4\varepsilon.$$

They are distinct and in the displayed increasing order. Their sum is $5m$ and their median is 1. The first two squares are less than 1, and the last two are greater than 1, so the median of the squares is exactly 1.

This proves that the statement is true exactly when $m \leq 1/5$. In particular, if the statement is true, then $m \leq 1/5 < 1$, so the proposed condition $m < 1$ is necessary.

It is not sufficient. The five distinct numbers

$$-1, \quad 0, \quad 1, \quad \frac{5}{4}, \quad \frac{7}{4}$$

have mean $3/5 < 1$ and median 1. Their squares in increasing order are

$$0, \quad 1, \quad 1, \quad \frac{25}{16}, \quad \frac{49}{16},$$

whose median is 1, not strictly greater than 1. Thus the statement is false for $m = 3/5$ despite the proposed condition being true.

The condition is necessary but not sufficient, giving **A**.

**6. F**

At the start of any cycle, let the dye proportions in $A$ and $B$ be $a$ and $b$. After the outward transfer, the proportion in $B$ is

$$\frac{2b + ta}{2 + t}.$$

After the return transfer, the new proportions in $A$ and $B$ are therefore

$$a' = \frac{(2 - t)a + 2tb}{2 + t}, \qquad b' = \frac{ta + 2b}{2 + t}.$$

Subtracting gives

$$a' - b' = r(a - b), \qquad r = \frac{2(1 - t)}{2 + t}.$$

Since $0 < t < 1$, we have $0 < r < 1$. Initially the difference in proportions is 1, so after $n$ cycles it is $r^n$. The total amount of dye remains 1 litre, and the tank volumes return to 1 and 2 litres after each cycle. Thus

$$c_n + 2b_n = 1, \qquad c_n - b_n = r^n,$$

which gives $c_n = (1 + 2r^n)/3$.

The given ratio becomes

$$5(1 + 2r) = 8(1 + 2r^3),$$

and hence

$$16r^3 - 10r + 3 = (2r - 1)(8r^2 + 4r - 3) = 0.$$

Exactly two of its roots satisfy $0 < r < 1$:

$$r = \frac{1}{2}, \qquad r = \frac{\sqrt{7} - 1}{4}.$$

The other root is negative. Solving the definition of $r$ for $t$ gives

$$t = \frac{2(1 - r)}{2 + r}.$$

The two admissible values of $t$ are consequently

$$\frac{2}{5}, \qquad \frac{2(5 - \sqrt{7})}{7 + \sqrt{7}} = 2 - \frac{4\sqrt{7}}{7}.$$

Both lie strictly between 0 and 1, and the formula is one-to-one for $0 < r < 1$, so there are no further possibilities. Their sum is

$$\boxed{\frac{12}{5} - \frac{4\sqrt{7}}{7}}.$$

**7. B**

The given sum and product show that $f(x)$ and $f(-x)$ are the roots of

$$z^2 - (2x^2 + 2)z + x^4 + x^2 + 1 = 0.$$

Thus, as an unordered pair,

$$\{f(x), f(-x)\} = \{x^2 + x + 1,\ x^2 - x + 1\}.$$

For $x \neq 0$, the two displayed values are different; either may be assigned to $f(x)$, provided the other is assigned to $f(-x)$. At $x = 0$, the equations force $f(0) = 1$.

Statement I is false. For example, take $f(x) = x^2 + x + 1$ except that the values at 1 and $-1$ are interchanged, so $f(1) = 1$ and $f(-1) = 3$. This preserves both equations. The values at $0, 2, -2$ force any quadratic agreeing there to be $x^2 + x + 1$, which does not agree with this $f$ at 1.

For every $x$,

$$f(x) \geq x^2 + 1 - |x| = \left(|x| - \frac{1}{2}\right)^2 + \frac{3}{4}.$$

One of $f(1/2)$ and $f(-1/2)$ is $3/4$, so this lower bound is attained. Statement II is true.

For statement III, compare $f$ with the fixed quadratic $g(x) = x^2 + x + 1$. On each pair of inputs $x, -x$ with $x > 0$, the functions $f$ and $g$ produce the same two values, possibly in the opposite order. They also agree at 0. Thus, for any fixed $y$, the equations $f(x) = y$ and $g(x) = y$ have the same number of distinct real solutions. The equation

$$x^2 + x + 1 = y$$

has discriminant $4y - 3$, which is positive for every $y > 3/4$. It then has exactly two distinct real solutions. Statement III is true.

Exactly II and III must hold. No continuity or consistent choice of a quadratic branch has been assumed.

**8. B**

Choose perpendicular coordinate axes along $PA$ and $PB$, with $P$ as the origin. The angle conditions allow us to write

$$A = (a, 0), \qquad B = (0, b), \qquad C = (-t, -t),$$

where $a, b, t > 0$. The three given side lengths give

$$a^2 + b^2 = 10, \qquad a^2 + 2at + 2t^2 = 5, \qquad b^2 + 2bt + 2t^2 = 17.$$

Put $L = a + b + 2t$, which is positive. Adding the last two equations and subtracting the first gives $2tL = 12$. Subtracting the second equation from the third gives $(b - a)L = 12$. Hence

$$b - a = 2t > 0.$$

Let $u = a + b$ and $v = b - a = 2t$. We have

$$u^2 + v^2 = 20, \qquad v(u + v) = 12.$$

Also $0 < v < u$, since $a > 0$, so $v^2 < 10$. Using $uv = 12 - v^2$,

$$v^2(20 - v^2) = (12 - v^2)^2,$$

which simplifies to

$$(v^2 - 4)(v^2 - 18) = 0.$$

The value $v^2 = 18$ is impossible because $v^2 < 10$. Thus $v = 2$, $u = 4$, and

$$a = 1, \qquad b = 3, \qquad t = 1.$$

Consequently

$$PA + PB + PC = a + b + \sqrt{2}\, t = \boxed{4 + \sqrt{2}}.$$

These values are geometrically consistent: $A = (1, 0)$, $B = (0, 3)$ and $C = (-1, -1)$ have the stated side lengths and angles at the origin. The origin is strictly inside their triangle, since it is $(3A + B + 3C)/7$.

**9. D**

When $c = p/q$ and $d = b$, polynomial division gives

$$P(n) = \left(\frac{pn}{q} + b\right)(n^2 + 1), \qquad \frac{P(n)}{n^2+1} = \frac{pn}{q} + b.$$

To test sufficiency, take

$$p = 2, \quad q = 3, \quad b = d = \frac{1}{2}, \quad c = \frac{2}{3}.$$

These satisfy all the hypotheses, and the proposed condition holds because $pb = 1$. However,

$$\frac{P(n)}{n^2+1} = \frac{4n+3}{6}$$

is never an integer: its numerator is odd, whereas every multiple of 6 is even. Thus the condition is not sufficient.

To test necessity, instead take

$$p = 2, \quad q = 3, \quad b = d = \frac{1}{3}, \quad c = \frac{2}{3}.$$

For every integer $j \geq 0$, the positive integer $n = 1 + 3j$ gives

$$\frac{P(n)}{n^2+1} = \frac{2n+1}{3} = 1 + 2j,$$

which is an integer. There are infinitely many such $n$, so the statement is true. But $pb = 2/3$ is not an integer, and the proposed condition is false. Thus the condition is not necessary.

The condition is neither necessary nor sufficient, giving **D**.

**10. C**

Place $A = (-1, 0)$, $B = (1, 0)$ and $C = (0, \sqrt{3})$, and write $P = (x, y)$. The sides $BC$ and $CA$ make angles of $60^\circ$ with $AB$. The distances to the three side lines are

$$d_1 = y, \qquad d_2 = \frac{\sqrt{3} - y - \sqrt{3}x}{2}, \qquad d_3 = \frac{\sqrt{3} - y + \sqrt{3}x}{2}.$$

For example, either sloping-side distance is the corresponding vertical separation multiplied by $\cos 60^\circ = 1/2$.

Expanding and completing the square gives

$$d_1^2 + d_2^2 + d_3^2 = 1 + \frac{3}{2}\left(x^2 + \left(y - \frac{\sqrt{3}}{3}\right)^2\right).$$

Thus the required inequality describes the circle and its interior with centre

$$O = \left(0, \frac{\sqrt{3}}{3}\right), \qquad r = \frac{\sqrt{3}}{3}.$$

This is the incircle of the triangle, so its whole interior is within $ABC$.

The conditioning region $y \geq \sqrt{3}/2$ is the top triangle of half the original side length. Its area is therefore $\sqrt{3}/4$. Its lower boundary is a distance

$$\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3} = \frac{r}{2}$$

above $O$. The favourable region is the smaller circular segment above this boundary. The radii to its endpoints make a central angle $2\pi/3$, since each makes an angle $\pi/3$ with the upward radius. Its area is

$$\frac{1}{2}r^2\frac{2\pi}{3} - \frac{1}{2}r^2 \sin\frac{2\pi}{3} = \frac{\pi}{9} - \frac{\sqrt{3}}{12}.$$

Dividing by the area of the conditioning region gives

$$\boxed{\frac{\pi/9 - \sqrt{3}/12}{\sqrt{3}/4} = \frac{4\pi}{9\sqrt{3}} - \frac{1}{3}}.$$

**11. C**

Lines I and II are valid. For example, independence of $A$ and $B$ gives

$$\Pr(A \text{ and } B) = \frac{4}{9},$$

so the probability that neither occurs is $1 - 2/3 - 2/3 + 4/9 = 1/9$.

Line III is the first invalid deduction: independence of each pair does not imply independence of all three events together.

To obtain the sharp range, let $t$ be the probability that all three events occur. The probability of each specified pattern with exactly two events occurring is $4/9 - t$. The probability that only $A$ occurs is

$$\frac{2}{3} - \left(\frac{4}{9} - t\right) - \left(\frac{4}{9} - t\right) - t = t - \frac{2}{9},$$

and the same holds for only $B$ or only $C$. The probability that none occurs is consequently

$$1 - t - 3\left(\frac{4}{9} - t\right) - 3\left(t - \frac{2}{9}\right) = \frac{1}{3} - t.$$

Non-negativity of the probabilities for exactly one event and for no events requires

$$\frac{2}{9} \le t \le \frac{1}{3}.$$

Every $t$ satisfying these inequalities is possible: assign probabilities $t$, $4/9 - t$, $t - 2/9$ and $1/3 - t$ to the all-three pattern, each of the three exactly-two patterns, each of the three exactly-one patterns, and the none pattern, respectively. These eight probabilities are non-negative, sum to 1, and give the required individual and pair probabilities.

Finally,

$$u = 1 - \left(\frac{1}{3} - t\right) = \frac{2}{3} + t,$$

so the complete set of possible values is

$$\boxed{\frac{8}{9} \le u \le 1}.$$

Thus **C** is correct.

**12. F**

Write

$$f(x) = a\left(x - \frac{1}{2}\right)^2 + b\left(x - \frac{1}{2}\right) + c,$$

so that $f(1/2) = c$. Reflection replaces the middle term by its negative:

$$f(1-x) = a\left(x - \frac{1}{2}\right)^2 - b\left(x - \frac{1}{2}\right) + c.$$

The odd powers of $x - 1/2$ have integral zero from 0 to 1, while

$$\int_0^1 \left(x - \frac{1}{2}\right)^2 \mathrm{d}x = \frac{1}{12}, \qquad \int_0^1 \left(x - \frac{1}{2}\right)^4 \mathrm{d}x = \frac{1}{80}.$$

Expanding the two integrands therefore gives

$$\frac{a^2}{80} + \frac{b^2 + 2ac}{12} + c^2 = 1, \qquad \frac{a^2}{80} + \frac{2ac - b^2}{12} + c^2 = 0.$$

Adding and halving yields

$$\frac{a^2}{80} + \frac{ac}{6} + c^2 = \frac{1}{2}.$$

Complete the square:

$$\frac{1}{80}\left(a + \frac{20c}{3}\right)^2 + \frac{4c^2}{9} = \frac{1}{2}.$$

Hence $c^2 \leq 9/8$, and in particular $c \leq 3\sqrt{2}/4$.

Equality is attainable. Set

$$c = \frac{3\sqrt{2}}{4}, \qquad a = -5\sqrt{2}, \qquad b = \sqrt{6}.$$

The squared term vanishes, and $b^2/12 = 1/2$, so the two original integrals are 1 and 0, respectively. The polynomial is genuinely quadratic because $a \neq 0$. The greatest value is therefore

$$\boxed{\frac{3\sqrt{2}}{4}}.$$

**13. E**

For statement I, fix $x$ with $0 < x < 1$, and put

$$y = \frac{1-x}{x} > 0.$$

For every integer $n \geq 2$, the binomial expansion gives

$$x^{-n} = (1+y)^n \geq \frac{n(n-1)}{2} y^2.$$

Consequently

$$F_n(x) = nx^n(1-x) \leq \frac{2(1-x)}{(n-1)y^2}.$$

Choose a positive integer $N$ with

$$N > 1 + \frac{8(1-x)}{y^2}.$$

Then the right-hand side is less than $1/4$ for every $n \geq N$. This proves I. Notice that this choice of $N$ depends on $x$.

For statement III, fix any positive integer $n$ and choose

$$x = 1 - \frac{1}{2n}.$$

This satisfies $0 < x < 1$. The finite geometric-series identity gives

$$1 - x^n = (1-x)(1 + x + \cdots + x^{n-1}) \leq n(1-x) = \frac{1}{2}.$$

Thus $x^n \geq 1/2$, and

$$F_n(x) = nx^n(1-x) = \frac{x^n}{2} \geq \frac{1}{4}.$$

This proves III, including the equality case $n = 1$, $x = 1/2$.

Statement II is false: for any proposed $N$, take $n = N$ and the value of $x$ just constructed. It gives $F_N(x) \geq 1/4$.

Exactly I and III are true, so the answer is **E**.

**14. A**

Put $s = \sin\theta$ and $c = \cos\theta$, so $s^2 + c^2 = 1$. Using $\cos v = \sin(\pi/2 - v)$ and the two branches of the sine equation, the given equation is equivalent to at least one of

$$3s - c = 1 + 4k, \qquad s + 3c = 1 + 4k,$$

where $k$ is an integer.

The identity

$$(3s - c)^2 + (s + 3c)^2 = 10$$

shows that each linear expression has absolute value at most $\sqrt{10}$. The only possible right-hand sides are therefore $-3$ and 1. Each of these has absolute value strictly less than $\sqrt{10}$, so each of the four lines meets the unit circle twice.

First add the $s$-coordinates with both families counted, then correct for any shared points. For the two points on $3s - c = t$, substituting $c = 3s - t$ into $s^2 + c^2 = 1$ gives

$$10s^2 - 6ts + t^2 - 1 = 0.$$

Their $s$-coordinates sum to $3t/5$. For the two points on $s + 3c = t$, the corresponding equation is

$$10s^2 - 2ts + t^2 - 9 = 0,$$

so their $s$-coordinates sum to $t/5$. These sums follow by comparing coefficients after writing each quadratic as a product of its two linear factors. The sum before correcting for shared points is

$$\frac{4}{5}(-3 + 1) = -\frac{8}{5}.$$

At a shared point, both $U = 3s - c$ and $V = s + 3c$ are either $-3$ or 1, and $U^2 + V^2 = 10$. Therefore the only possible pairs are

$$(U, V) = (-3, 1) \qquad \text{or} \qquad (U, V) = (1, -3).$$

Solving the two linear equations in each case gives the unit-circle points

$$(s, c) = \left(-\frac{4}{5}, \frac{3}{5}\right), \qquad (s, c) = (0, -1).$$

Both are valid shared points. Each was counted twice, so their $s$-coordinates must each be subtracted once. There are six distinct solution points, and the required sum is

$$-\frac{8}{5} - \left(-\frac{4}{5} + 0\right) = \boxed{-\frac{4}{5}}.$$

**15. E**

A simple real root of $P$ must also be a root of $Q$. Otherwise, close to this root, $P$ changes sign while $Q$ retains one nonzero sign, so their product is negative on one side. The same argument with $P$ and $Q$ interchanged shows that they have exactly the same real roots.

A real cubic with no repeated real root has either one or three real roots. If there were three, both monic cubics would be the product of the same three linear factors, contrary to $P \neq Q$. Hence there is exactly one common real root, say $r$, and

$$P(x) = (x - r)U(x), \qquad Q(x) = (x - r)V(x),$$

where $U$ and $V$ are monic quadratics with no real roots. Thus $U(x) > 0$ and $V(x) > 0$ for every real $x$.

The other intersection has some abscissa $s \neq r$, and its ordinate is $(s - r)U(s) \neq 0$. This proves I. The gradient of the line joining the intersections is

$$\frac{(s - r)U(s) - 0}{s - r} = U(s) > 0,$$

which proves II.

For a counterexample to III, take

$$P(x) = (x - 2)(x^2 + x + 1), \qquad Q(x) = (x - 2)(x^2 - x + 1).$$

The quadratic factors are positive for every real $x$, so all the hypotheses hold. Since

$$P(x) - Q(x) = 2x(x - 2),$$

the intersections have abscissae 0 and 2. At $x = 0$, the gradients are $P'(0) = -1$ and $Q'(0) = 3$. Thus III is false.

Therefore precisely I and II must be true: **E**.

**16. B**

Write $p_A, p_B, p_C, p_D$ for the four head probabilities. On expanding the polynomial

$$F(x) = \prod_{J=A,B,C,D} (1 - p_J + p_J x),$$

the coefficient of $x^j$ is the probability of exactly $j$ heads: choosing $p_J x$ from a factor means that coin $J$ is heads. The given equalities therefore imply

$$F(x) = x^4 F(1/x) \qquad (x \neq 0).$$

Since $p_A = 1/3$, the factor for $A$ is $(x+2)/3$, so $F(-2) = 0$. The displayed identity gives $F(-1/2) = 0$. Hence one of the other factors vanishes at $-1/2$, and that coin has head probability $2/3$. Relabel the other three coins so that it is $B$.

Let the remaining head probabilities be $q$ and $r$. Equality of the probabilities of zero and four heads gives

$$\frac{2}{9}(1-q)(1-r) = \frac{2}{9}qr,$$

so $r = 1 - q$. Put $u = q(1-q)$; then $0 < u \leq 1/4$.

For the pair $A, B$, the probabilities of zero, one and two heads are $2/9, 5/9, 2/9$. For the pair $C, D$, they are $u, 1-2u, u$. Thus

$$\Pr(\text{exactly two heads}) = \frac{2}{9}u + \frac{5}{9}(1-2u) + \frac{2}{9}u = \frac{5-6u}{9}.$$

Also,

$$\Pr(A \text{ is heads and exactly two heads}) = \frac{1}{3}\left[\frac{2}{3}u + \frac{1}{3}(1-2u)\right] = \frac{1}{9}.$$

The required conditional probability is therefore $1/(5-6u)$. This increases as $u$ increases, so its greatest value is

$$\frac{1}{5-6(1/4)} = \boxed{\frac{2}{7}}.$$

It is attained when $q = r = 1/2$. The probabilities $1/3, 2/3, 1/2, 1/2$ satisfy both original symmetry conditions, so the bound is achievable. The answer is **B**.

**17. D**

The product conditions give

$$b = \frac{1}{a}, \qquad c = 2a, \qquad d = \frac{3}{a}.$$

Set

$$x = a + c = 3a, \qquad y = b + d = \frac{4}{a}.$$

Then $x > 0$, $y > 0$, $xy = 12$, and

$$a + b + c + d = x + y, \qquad ac + bd = \frac{2}{9}x^2 + \frac{3}{16}y^2.$$

Suppose S and T both hold. Since $x + y$ is rational,

$$x^2 + y^2 = (x + y)^2 - 24$$

is rational. Combining this with the rational value of $\frac{2}{9}x^2 + \frac{3}{16}y^2$, and using

$$\frac{2}{9} - \frac{3}{16} = \frac{5}{144} \neq 0,$$

shows that $x^2$ and $y^2$ are each rational. Now

$$x - y = \frac{x^2 - y^2}{x + y}$$

is rational, since $x + y > 0$. Hence $x, y$, and consequently $a = x/3$, $b = y/4$, $c = 2x/3$ and $d = 3y/4$, are all rational.

S alone is not sufficient. Take

$$a = \frac{9 + \sqrt{33}}{6}, \qquad b = \frac{1}{a}, \qquad c = 2a, \qquad d = \frac{3}{a}.$$

Here $3a^2 - 9a + 4 = 0$, so $a + b + c + d = 3a + 4/a = 9$, but $a$ is irrational.

T alone is not sufficient either. Take $a = \sqrt{2}$ and again $b = 1/a$, $c = 2a$, $d = 3/a$. Then

$$ac + bd = 2a^2 + \frac{3}{a^2} = \frac{11}{2},$$

but $a$ is irrational.

Thus the answer is **D**.

**18. A**

The speed is

$$\sqrt{(1-t^2)^2+4t^2} = \sqrt{(1+t^2)^2} = 1+t^2.$$

Consequently,

$$D(T) = \int_0^T (1+t^2)\,\mathrm{d}t = T + \frac{T^3}{3}.$$

Integrating the two coordinate rates and using the initial position gives

$$x(T) = T - \frac{T^3}{3}, \qquad y(T) = T^2.$$

In particular $R(T) > 0$ for $T > 0$. Put $u = T^2/3 > 0$. Then

$$\left(\frac{D(T)}{R(T)}\right)^2 = \frac{(1+u)^2}{(1-u)^2+3u} = 1 + \frac{u}{u^2+u+1}.$$

Since $(u-1)^2 \geq 0$, we have $u^2+u+1 \geq 3u$, and therefore

$$\left(\frac{D(T)}{R(T)}\right)^2 \leq \frac{4}{3}.$$

Equality holds at $u = 1$, or $T = \sqrt{3}$, so the greatest ratio is

$$\boxed{\frac{2}{\sqrt{3}}}.$$

The answer is **A**.

**19. C**

To prove necessity, suppose that $n \geq 5$. Since $P(1) = 1$, the expressions

$$U(x) = \frac{P(x) - x}{x - 1}, \qquad V(x) = \frac{P(x) - x^2}{x - 1}$$

are quadratic polynomials. They have the same nonzero leading coefficient, and

$$U(x) - V(x) = x,$$

so they also have the same constant term.

Each of $2, 3, 4, 5$ must be a root of $U$ or $V$. None can be a root of both, since $U(k) - V(k) = k \neq 0$. A quadratic has at most two distinct roots, so these four numbers must split into two pairs: the roots of $U$ and the roots of $V$.

The two pairs must have equal products, because the quadratics have the same leading coefficient and constant term. This is impossible: the three possible divisions give products

$$6 \text{ and } 20, \qquad 8 \text{ and } 15, \qquad 10 \text{ and } 12.$$

Thus the existence statement implies $n \leq 4$, proving necessity.

To prove sufficiency, take

$$P(x) = x + (x - 1)(x - 3)(x - 4).$$

This has degree exactly 3, and

$$P(1) = 1, \quad P(2) = 4, \quad P(3) = 3, \quad P(4) = 4.$$

Every displayed value is of the required form. The same polynomial therefore works for every positive integer $n \leq 4$, proving sufficiency.

The condition is both necessary and sufficient, giving **C**.

**20. C**

Place the centre of the rectangle at the origin and take

$$A = (-\sqrt{3}, -1), \quad B = (\sqrt{3}, -1), \quad C = (\sqrt{3}, 1), \quad D = (-\sqrt{3}, 1).$$

Write $P = (x, y)$. The given condition becomes

$$PA^2 + PC^2 = 2(x^2 + y^2 + 4) = 12,$$

so $x^2 + y^2 = 2$.

Let $K = PA \cdot PB \cdot PC \cdot PD$. Pairing opposite vertices gives

$$PA^2 PC^2 = 36 - (2\sqrt{3}x + 2y)^2,$$
$$PB^2 PD^2 = 36 - (2\sqrt{3}x - 2y)^2.$$

Multiplying these expressions as a difference of two squares, then using $y^2 = 2 - x^2$, yields

$$\begin{aligned} K^2 &= (36 - 12x^2 - 4y^2)^2 - 192x^2y^2 \\ &= (28 - 8x^2)^2 - 192x^2(2 - x^2) \\ &= 256x^4 - 832x^2 + 784 \\ &= 4(8x^2 - 13)^2 + 108. \end{aligned}$$

Hence $K^2 \geq 108$, so $K \geq 6\sqrt{3}$.

Equality is attainable: choose $x^2 = 13/8$ and $y^2 = 3/8$, which satisfy $x^2 + y^2 = 2$. Thus the least product is $\boxed{6\sqrt{3}}$, giving **C**.