1. The positive integers occurring in either of the sequences

$$1, 4, 7, 10, \ldots \qquad \text{and} \qquad 1, 2, 4, 8, \ldots$$

are combined into one increasing sequence $a_1, a_2, a_3, \ldots$. An integer occurring in both original sequences is listed only once.

What is the sum of all the indices n satisfying

$$1 \leq n \leq 99 \qquad \text{and} \qquad a_{n+1} - a_n = 1\,?$$

A 4

B 58

C 64

D 68

E 166

F 170

2. A sequence of real numbers satisfies

$$u_{n+1} = u_n^2 - 3 \qquad (n = 1, 2, 3, \ldots).$$

Its first three terms form an arithmetic progression with nonzero common difference.

What is the sum of all the distinct possible values of u_4?

A 11

B 53

C 74

D 101

E 95

F 382

3. Five distinct positive integers are written in increasing order.

When the smallest is removed, the mean of the four remaining integers equals the median of the original five.

When the largest is instead removed from the original five, the mean of the four remaining integers equals their median.

What is the smallest possible sum of the original five integers?

A 5

B 16

C 17

D 20

E 21

F 25

4. A real number $t > 2$ and integers a, b satisfy

$$t^3 - 2t = a, \qquad t^2 - at + b = 2.$$

What is the smallest possible value of a?

A 4

B 5

C 9

D 18

E 21

F 56

5. For nonzero real x, define

$$f(x) = x^2 + \frac{1}{x^2}, \qquad g(x) = x^2 - 3x.$$

What is the sum of the squares of all the distinct nonzero real numbers a for which there is a nonzero real number $b \neq a$ satisfying

$$f(a) = f(b) \qquad \text{and} \qquad g(a) = g(b)\,?$$

A 4

B 7

C 9

D 11

E 14

F 18

6. For real x with $x \neq 0$ and $x \neq -1$, what is the least possible value of

$$x^2 + (x+1)^2 + \frac{1}{2x^2} + \frac{1}{2(x+1)^2}\,?$$

A $\frac{9}{2}$

B $2\sqrt{2}$

C 4

D 5

E $\frac{45}{8}$

F 6

7. For $r = 0, 1, \ldots, 49$, let a_r be the coefficient of x^r in

$$(1+x)^{47}(1-x)^2.$$

These coefficients satisfy the identity

$$a_r = \binom{49}{r} \frac{(r-21)(r-28)}{588}.$$

What is the value of

$$\frac{|a_0| + |a_1| + \cdots + |a_{49}|}{\binom{47}{21}} \, ?$$

A 0

B $\frac{4}{9}$

C $\frac{8}{9}$

D 1

E $\frac{16}{9}$

F $\frac{2}{9}$

8. A rectangular sheet $ABCD$ is folded along the diagonal AC, so that the two triangular parts lie in different planes. In the folded position,

$$AB = BD = 1,$$

and the directions of AB and CD are perpendicular.

What is the volume of the tetrahedron with vertices A, B, C, D?

The volume of a tetrahedron is one third of the area of a triangular face multiplied by the perpendicular distance of the opposite vertex from that face.

A $\frac{1}{12}$

B $\frac{\sqrt{2}}{12}$

C $\frac{\sqrt{3}}{12}$

D $\frac{1}{6}$

E $\frac{\sqrt{2}}{6}$

F $\frac{1}{3}$

9. For a real constant k, the polynomial

$$p(t) = t^3 - kt^2 + (k-1)t + \frac{k}{32}$$

has $\sin\theta$ and $\cos\theta$ as roots, where $0 < \theta < \frac{\pi}{4}$.

What is the value of k?

A $\frac{7}{8}$

B $\frac{9}{8}$

C 1

D $\frac{5}{4}$

E $\frac{1+\sqrt{2}}{2}$

F $\frac{9}{4}$

10. A circle meets the parabola $y = x^2$ at exactly three distinct points. Their x-coordinates, in increasing order, form an arithmetic progression with common difference 2.

What is the sum of all the distinct possible values of the area of the circle?

A $\dfrac{25\pi}{4}$

B 10π

C 20π

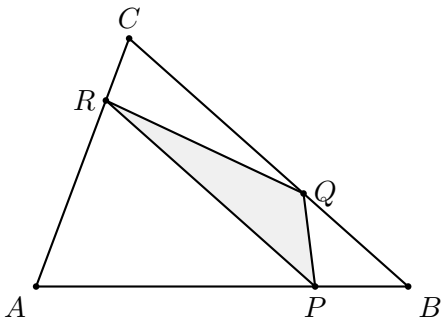
D $\dfrac{105\pi}{4}$

E $\dfrac{65\pi}{4}$

F $\dfrac{65\pi}{2}$

11. Three points P, Q, R start at the vertices A, B, C, respectively, of a triangle. They travel along AB, BC, CA, respectively, at constant speeds, reaching the other endpoint after 1, 2 and 3 seconds, respectively. Each point stops when it reaches that endpoint.

For how long during the first 3 seconds is the area of triangle PQR at most one third of the area of triangle ABC?



One possible position during the motion is shown.

A $\left(\sqrt{2} - \frac{1}{2}\right)$ seconds

B $\frac{5}{6}$ seconds

C $(\sqrt{2} - 1)$ seconds

D $\sqrt{2}$ seconds

E $\frac{3}{2}$ seconds

F $(2 - \sqrt{2})$ seconds

12. For each real angle θ, let m and M be the smallest and largest, respectively, of the four numbers

$$|\sin\theta|, \qquad |\cos\theta|, \qquad |\sin\theta+\cos\theta|, \qquad |\sin\theta-\cos\theta|.$$

What is the greatest possible value of mM?

A $\frac{1}{3}$

B $\frac{2}{5}$

C $\frac{3}{5}$

D $\frac{1}{2}$

E $\frac{3}{4}$

F 1

13. Three consecutive vertices A, B, C of a parallelogram $ABCD$ lie on the circle

$$x^2 + y^2 = 1.$$

The fourth vertex satisfies $OD = \sqrt{5}$, where O is the origin.

If the area of the parallelogram is K, what is the greatest possible value of K^2?

A $2 + \sqrt{5}$

B 4

C 6

D $\dfrac{27}{4}$

E 8

F $2 + 2\sqrt{5}$

14. A rectangle has sides parallel to the coordinate axes. One vertex is $(-9, -5)$, and the opposite vertex lies on the circle

$$x^2 + y^2 = 25.$$

What is the greatest possible area of the rectangle?

A 90

B $\dfrac{115}{2} + 7\sqrt{34}$

C $\dfrac{115}{2} + 35\sqrt{2}$

D 108

E $\dfrac{123}{2} + 35\sqrt{2}$

F 140

15. A cubic polynomial f satisfies

$$f(0) = 0, \qquad f(1) = 1, \qquad f'(0)f'(1) = 4.$$

It is also given that $f'(x) \geq 0$ whenever $0 \leq x \leq 1$.

What is the greatest possible value of $f(1/2)$?

A $\frac{1}{8}$

B $\frac{1}{2}$

C $\frac{5}{8}$

D $\frac{7}{8}$

E 1

F There is no greatest value.

16. For a real constant h, consider the curves

$$C_1: \quad y = \frac{x^3}{3} - \frac{x}{2}, \qquad C_2: \quad y = \frac{(x-2)^3}{3} - \frac{x-2}{2} + h.$$

For each value of h for which the curves meet and their tangents are perpendicular at an intersection, the area enclosed by the two curves is calculated.

What is the sum of the squares of the distinct areas obtained?

A $\frac{8}{9}$

B 28

C 56

D $\frac{1000}{9}$

E 112

F 224

17. Six counters initially carry the numbers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 7.$$

At each stage, one pair of the counters currently present is chosen uniformly at random. The chosen counters are removed and replaced by one counter carrying their sum. The process stops when two counters remain.

Given that the final two numbers are equal, what is the probability that the first pair chosen carried 1 and 2?

A $\frac{1}{15}$

B $\frac{1}{14}$

C $\frac{1}{16}$

D $\frac{4}{49}$

E $\frac{4}{7}$

F $\frac{1}{28}$

18. Nine cards are numbered $1, 2, \ldots, 9$. Three are drawn without replacement, with every selection of three cards equally likely.

After seeing these three cards, a player must discard either one or two of them. The discarded cards are not returned to the pack. The same number of cards is then drawn from the six undrawn cards, without replacement, with every selection of that size equally likely.

The player wins if the sum of the final three numbers is divisible by 3. What is the greatest possible probability of winning, when the player chooses which cards to discard as favourably as possible?

A $\frac{3}{7}$

B $\frac{27}{70}$

C $\frac{9}{20}$

D $\frac{1}{2}$

E $\frac{3}{5}$

F $\frac{19}{28}$

19. Five cards are drawn successively without replacement from nine cards numbered $1, 2, \ldots, 9$. Each remaining card is equally likely at each draw. The numbers, in order, are y_0, y_1, y_2, y_3, y_4.

It is given that there exists a polynomial f of degree at most 2 such that

$$f(k) = y_k \qquad (k = 0, 1, 2, 3, 4).$$

What is the probability that f has degree exactly 2?

A $\dfrac{1}{1890}$

B $\dfrac{1}{7}$

C $\dfrac{1}{4}$

D $\dfrac{1}{2}$

E $\dfrac{3}{5}$

F $\dfrac{2}{5}$

20. For $a \geq 0$, let

$$f_a(x) = x^3 + ax(1-x).$$

The trapezium rule is used to estimate

$$\int_0^1 f_a(x)\,\mathrm{d}x$$

with two strips whose boundaries are $x = 0$, $x = t$ and $x = 1$, where $0 < t < 1$. The value of t may be chosen separately for each a.

Let m and M be the smallest and largest values of a for which this estimate can be exact. What is mM?

A $3 - \dfrac{\sqrt{3}}{2}$

B $\dfrac{9}{4}$

C $3 + \dfrac{\sqrt{3}}{2}$

D $\dfrac{15 - \sqrt{3}}{6}$

E 3

F $\dfrac{21}{4} - 2\sqrt{3}$