1. C

The terms of the first sequence leave remainder 1 when divided by 3. Successive powers of 2 leave remainders $1, 2, 1, 2, \ldots$, since multiplying a remainder by 2 changes 1 to 2 and 2 to 1.

Thus the even powers of 2 are already present. The extra terms are

$$2, \quad 8, \quad 32, \quad 128, \quad 512, \ldots.$$

Each extra term lies one unit after a term of the arithmetic sequence and two units before its next term. Consequently a gap of 1 occurs precisely just before an extra term.

There are already 100 terms of the arithmetic sequence no greater than 298, so $a_{100} \leq 298$. The only relevant extra terms are therefore $2, 8, 32, 128$. Their predecessors are $1, 7, 31, 127$.

The ranks of those predecessors in the arithmetic sequence are $1, 3, 11, 43$. Respectively $0, 1, 2, 3$ earlier extra terms have been inserted. Their ranks in the combined sequence are therefore

$$1, \quad 4, \quad 13, \quad 46.$$

All four ranks satisfy $1 \leq n \leq 99$, and no later extra term occurs among the first 100 entries. The required sum is

$$1 + 4 + 13 + 46 = \boxed{64}.$$

2. E

The recurrence gives

$$u_3 - u_2 = (u_2^2 - 3) - (u_1^2 - 3) = (u_2 - u_1)(u_1 + u_2).$$

The first three terms form an arithmetic progression, so $u_3 - u_2 = u_2 - u_1 \neq 0$. Division by the nonzero difference gives $u_1 + u_2 = 1$.

Put $u_1 = t$. Since $u_2 = t^2 - 3$, we obtain

$$t^2 + t - 4 = 0, \qquad u_2 = 1 - t, \qquad u_3 = 2 - 3t.$$

Hence

$$u_4 = (2 - 3t)^2 - 3 = 1 - 12t + 9t^2 = 37 - 21t.$$

The two roots of $t^2 + t - 4 = 0$ are distinct and real, and neither is $1/2$. Each therefore gives a nonzero difference $u_2 - u_1 = 1 - 2t$ and satisfies the required arithmetic-progression condition. The two resulting fourth terms are distinct because $37 - 21t$ is one-to-one.

The two values of t sum to -1. The requested sum is consequently

$$2 \cdot 37 - 21(-1) = \boxed{95}.$$

3. E

Write the five integers as $a < b < c < d < e$. The two conditions give

$$b + c + d + e = 4c, \qquad a + b + c + d = 2(b + c).$$

Thus

$$d = b + c - a, \qquad e = 2c - 2b + a.$$

Put $u = b - a$ and $v = c - b$. Both are positive integers, and the five numbers become

$$a, \qquad a + u, \qquad a + u + v, \qquad a + 2u + v, \qquad a + 2v.$$

The requirement $e > d$ gives $v > 2u$. Hence $a \geq 1$, $u \geq 1$ and $v \geq 2u + 1 \geq 3$.

Their sum is

$$5a + 4u + 4v \geq 5 + 4 + 12 = 21.$$

Equality is attained by $a = 1$, $u = 1$ and $v = 3$, giving the five distinct positive integers

$$1, \quad 2, \quad 5, \quad 6, \quad 7.$$

After the first deletion the mean is 5, the original median. After the second deletion the mean and median are both $7/2$. Therefore the required minimum is $\boxed{21}$.

4. E

The second equation gives $t^2 = at - b + 2$. Substituting this twice into the first gives

$$\begin{aligned} a &= t(t^2 - 2) = t(at - b) = at^2 - bt \\ &= a(at - b + 2) - bt = (a^2 - b)t - a(b - 2). \end{aligned}$$

Consequently

$$(a^2 - b)t = a(b - 1).$$

Since $t > 2$, we have $a = t(t^2 - 2) > 4$. If $a^2 - b = 0$, the last identity would give $b = 1$ and then $a^2 = 1$, a contradiction. Therefore

$$t = \frac{a(b-1)}{a^2 - b}$$

is rational.

Write $t = m/n$ in lowest terms, where m, n are positive integers. The first equation gives

$$m^3 - 2mn^2 = an^3.$$

Thus n divides m^3. Since m and n have no common prime factor, $n = 1$. Hence t is an integer and $t \geq 3$.

It follows that

$$a = t(t^2 - 2) \geq 3(9 - 2) = 21.$$

Taking $t = 3$, $a = 21$ and $b = 56$ satisfies both original equations. The smallest possible value is therefore $\boxed{21}$.

5. F

The second equality gives

$$(a-b)(a+b-3)=0.$$

Since $a \neq b$, we have $a+b=3$. In particular,

$$a^2-b^2=(a-b)(a+b)=3(a-b)\neq 0.$$

Multiplying the first equality by $a^2b^2>0$ and rearranging gives

$$(a^2-b^2)(a^2b^2-1)=0.$$

It follows that $ab=1$ or $ab=-1$.

If $ab=1$, the numbers a,b are the two roots of $t^2-3t+1=0$. If $ab=-1$, they are the two roots of $t^2-3t-1=0$. Both quadratics have two distinct nonzero real roots, because their discriminants are 5 and 13 and their constant terms are nonzero. No root belongs to both quadratics, since the two polynomial expressions differ by 2.

Conversely, taking the two roots of either quadratic as a,b satisfies both original equalities. Hence exactly these four values of a are included. Their squared values sum to

$$(3^2-2\cdot 1)+(3^2-2\cdot(-1))=7+11=\boxed{18}.$$

6. A

Put $t = x(x+1)$. Completing the square gives

$$t = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4},$$

and the excluded inputs ensure $t \neq 0$.

Since $x^2 + (x+1)^2 = 2t + 1$, the expression in the question becomes

$$E = 2t + 1 + \frac{2t+1}{2t^2}.$$

Subtracting $9/2$ and factorising gives the exact identity

$$E - \frac{9}{2} = \frac{4t^3 - 7t^2 + 2t + 1}{2t^2} = \frac{(t-1)^2(4t+1)}{2t^2}.$$

Every factor has the required sign: the denominator is positive, the square is non-negative, and $4t + 1 \geq 0$. Hence $E \geq 9/2$ for every allowed x.

Equality is attained at $t = -1/4$, corresponding to $x = -1/2$. It is also attained at $t = 1$, corresponding to $x = (-1 \pm \sqrt{5})/2$. None of these inputs is excluded. Thus the least value is

$$\boxed{\frac{9}{2}}.$$

7. C

The given identity shows that the negative coefficients are precisely $a_{22}, a_{23}, \ldots, a_{27}$; $a_{21} = a_{28} = 0$.

The sum of all coefficients is the polynomial's value at $x = 1$, which is 0. The sum of their absolute values is therefore minus twice the sum of the negative coefficients.

Write $B_r = \binom{47}{r}$ for $0 \le r \le 47$, and put $B_r = 0$ for other integer values of r. Multiplication by $1 - 2x + x^2$ gives

$$a_r = B_r - 2B_{r-1} + B_{r-2}.$$

In the sum from $r = 22$ to $r = 27$, all the interior terms cancel:

$$\begin{aligned}\sum_{r=22}^{27} a_r &= B_{27} - B_{26} - B_{21} + B_{20} \\ &= 2(B_{20} - B_{21}),\end{aligned}$$

where the last equality uses $B_{27} = B_{20}$ and $B_{26} = B_{21}$.

Thus the required ratio is

$$\frac{4(B_{21} - B_{20})}{B_{21}} = 4\left(1 - \frac{21}{27}\right) = \boxed{\frac{8}{9}}.$$

8. D

Let $BC = AD = b$. Keep triangle ABC in a horizontal plane, with AB horizontal in that plane. Let H be the perpendicular projection of D onto this plane. Since CD has direction perpendicular to AB, and BC is also perpendicular to AB, the points B, C, H are collinear.

Two applications of Pythagoras give

$$AD^2 = AB^2 + BH^2 + DH^2 = AB^2 + BD^2 = 2.$$

Thus $b = \sqrt{2}$. Folding preserves $CD = AB = 1$, so $BD = CD = 1$. Subtracting

$$BH^2 + DH^2 = 1, \qquad CH^2 + DH^2 = 1$$

shows that H is the midpoint of BC. Hence

$$DH^2 = 1 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}.$$

The required volume is therefore

$$\frac{1}{3}\left(\frac{1}{2} \cdot 1 \cdot \sqrt{2}\right)\frac{1}{\sqrt{2}} = \boxed{\frac{1}{6}}.$$

The construction is feasible: take

$$A = (0, 0, 0),\ B = (1, 0, 0),\ C = (1, \sqrt{2}, 0) \text{ and } D = (1, 1/\sqrt{2}, 1/\sqrt{2}).$$

The two triangles have exactly the side lengths of the two halves of the stated rectangle.

9. B

Write $u = \sin\theta$, $v = \cos\theta$, $s = u + v$, and let w be the third root. Since $0 < \theta < \pi/4$, the first two roots are distinct and

$$1 < s < \sqrt{2}, \qquad uv = \frac{s^2 - 1}{2}.$$

Expanding $(t-u)(t-v)(t-w)$ and comparing coefficients gives

$$s + w = k, \qquad uv + sw = k - 1, \qquad uvw = -\frac{k}{32}.$$

The first two equalities imply

$$uv + (s-1)(w-1) = 0.$$

As $s > 1$, substitution of $uv = (s-1)(s+1)/2$ gives

$$w = \frac{1-s}{2}, \qquad k = \frac{s+1}{2} > 1.$$

Consequently $w = 1 - k$ and $uv = 2k(k-1)$. The product equation becomes

$$-2k(k-1)^2 = -\frac{k}{32}.$$

Since $k > 1$, this gives $k - 1 = 1/8$, so $k = \boxed{9/8}$.

Existence is not automatic from the necessary equations. For $k = 9/8$, the three roots are

$$\frac{5-\sqrt{7}}{8}, \qquad \frac{5+\sqrt{7}}{8}, \qquad -\frac{1}{8}.$$

The first two are positive, distinct, and have squares summing to 1, so they are $\sin\theta, \cos\theta$ for an angle satisfying $0 < \theta < \pi/4$.

10. E

Write the circle as

$$x^2 + y^2 + ux + vy + w = 0.$$

Its intersections with the parabola correspond to the real roots of

$$F(x) = x^4 + (1+v)x^2 + ux + w.$$

Let the three distinct roots be $a-2, a, a+2$. Dividing F by their three linear factors leaves a real linear factor. Since there are exactly three distinct real roots, the fourth root repeats one of these three.

The coefficient of x^3 is zero, so the sum of the four roots, counting the repeated root, is zero. The three cases give

$$4a - 2 = 0, \qquad 4a = 0, \qquad 4a + 2 = 0.$$

Thus $a = 1/2, 0, -1/2$.

If $a = 0$, then

$$F(x) = x^2(x-2)(x+2) = x^4 - 4x^2,$$

so the circle is $x^2 + y^2 - 5y = 0$, with radius squared $25/4$.

If $a = 1/2$, the repeated root is $-3/2$, and

$$F(x) = (x+3/2)^2(x-1/2)(x-5/2) = x^4 - \frac{11}{2}x^2 - 3x + \frac{45}{16}.$$

The circle is therefore

$$x^2 + y^2 - 3x - \frac{13}{2}y + \frac{45}{16} = 0,$$

whose centre is $(3/2, 13/4)$ and whose radius squared is

$$\frac{9}{4} + \frac{169}{16} - \frac{45}{16} = 10.$$

The case $a = -1/2$ is its reflection in the y-axis, and has the same area. Each displayed factorisation has exactly the required three distinct real roots, so every candidate is valid.

There are two distinct areas, $25\pi/4$ and 10π. Their sum is $\boxed{65\pi/4}$.

11. A

Let $p = AP/AB$, $q = BQ/BC$ and $r = CR/CA$, and let F be the area of PQR divided by the area of ABC. The three corner triangles have area fractions $p(1-r)$, $q(1-p)$ and $r(1-q)$, so

$$F = 1 - p - q - r + pq + qr + rp.$$

For $0 \le t \le 1$, the fractions are $p = t$, $q = t/2$, $r = t/3$. Hence

$$F = 1 - \frac{11t}{6} + t^2.$$

The condition $F \le 1/3$ is

$$6t^2 - 11t + 4 = (2t-1)(3t-4) \le 0.$$

Within this first time period it holds exactly when $1/2 \le t \le 1$.

For $1 \le t \le 2$, P has stopped, so $p = 1$ and

$$F = qr = \frac{t^2}{6}.$$

Here $F \le 1/3$ exactly when $1 \le t \le \sqrt{2}$.

For $2 \le t \le 3$, both P and Q have stopped, and $F = r = t/3 \ge 2/3$, so there are no further qualifying times. The complete qualifying period is $1/2 \le t \le \sqrt{2}$, with duration $\boxed{\sqrt{2} - 1/2}$ seconds.

12. C

Put $a = \min(|\sin\theta|, |\cos\theta|)$ and $b = \max(|\sin\theta|, |\cos\theta|)$. Then $0 \le a \le b$ and $a^2 + b^2 = 1$. Regardless of the signs of the sine and cosine, the four numbers, in some order, are

$$a, \quad b, \quad b + a, \quad b - a.$$

Thus $M = a + b$ and $m = \min(a, b - a)$. Put $u = a/b$, so $0 \le u \le 1$.

If $u \le 1/2$, then $m = a$, giving

$$mM = \frac{u(1+u)}{1+u^2} \le \frac{3}{5},$$

since this inequality is equivalent to

$$(2u - 1)(u + 3) \le 0.$$

If $u \ge 1/2$, then $m = b - a$, and

$$mM = \frac{1-u^2}{1+u^2} \le \frac{3}{5},$$

which is equivalent to $u^2 \ge 1/4$.

Both bounds are attained when $a = 1/\sqrt{5}$ and $b = 2/\sqrt{5}$. Hence the greatest value is $\boxed{3/5}$.

13. F

Let N be the midpoint of AC. It is also the midpoint of BD. If $N = O$, then $D = -B$, which would give $OD = 1$, so $N \neq O$.

Rotate the axes so that $N = (t, 0)$, where $0 < t < 1$. The chord endpoints may then be written

$$A = (t, h), \quad C = (t, -h), \qquad h = \sqrt{1 - t^2}.$$

Write $B = (u, v)$. Since the diagonals bisect one another, $D = (2t - u, -v)$. Subtracting $OB^2 = 1$ from $OD^2 = 5$ gives

$$4t^2 - 4tu = 4, \qquad u = t - \frac{1}{t}.$$

As B lies on the unit circle, $u \geq -1$. Therefore

$$t^2 + t - 1 \geq 0, \qquad t \geq \frac{\sqrt{5} - 1}{2}.$$

The distance of B from the line AC is $t - u = 1/t$. The parallelogram is twice the area of triangle ABC, so

$$K = 2h\frac{1}{t}, \qquad K^2 = 4\left(\frac{1}{t^2} - 1\right).$$

This decreases as positive t increases. Its greatest possible value is consequently

$$4\left(\frac{3 + \sqrt{5}}{2} - 1\right) = \boxed{2 + 2\sqrt{5}}.$$

Equality is feasible: take $t = (\sqrt{5}-1)/2$, $B = (-1, 0)$, $D = (\sqrt{5}, 0)$ and $A, C = (t, \pm\sqrt{1 - t^2})$. These points form the required non-degenerate parallelogram.

14. D

Let the opposite vertex be (x, y). On the circle, $x \geq -5 > -9$ and $y \geq -5$, so the area is

$$A = (x + 9)(y + 5).$$

To obtain a sharp bound, use $x^2 + y^2 = 25$ to check the identity

$$108 - A = (x - 3)^2 + (y - 4)^2 + \frac{1}{2}(x - y + 1)^2.$$

Every term on the right is non-negative, so $A \leq 108$. Equality is attained at $(x, y) = (3, 4)$, which lies on the circle and gives side lengths 12 and 9. The greatest area is therefore $\boxed{108}$.

15. D

Put $u = f'(0)$ and $v = f'(1)$. Both are positive and $uv = 4$. Since f' is a quadratic, it can be written

$$f'(x) = u(1-x)^2 + 2wx(1-x) + vx^2.$$

Integration from 0 to 1 gives $(u + w + v)/3 = 1$, so $w = 3 - u - v$.

Also,

$$f'(x) = \left(\sqrt{u}(1-x) - \sqrt{v}\,x\right)^2 + 2(w+2)x(1-x).$$

At $x = \sqrt{u}/(\sqrt{u} + \sqrt{v})$ the squared term is zero. Non-negativity therefore requires $w \geq -2$. Conversely, this condition makes both terms non-negative for $0 \leq x \leq 1$. Thus $u + v \leq 5$.

Since $uv = 4$,

$$(u-v)^2 = (u+v)^2 - 16 \leq 9.$$

Integrating the displayed quadratic from 0 to $1/2$ gives

$$f(1/2) = \frac{1}{2} + \frac{u-v}{8} \leq \frac{7}{8}.$$

Equality is attained by

$$f(x) = 3x^3 - 6x^2 + 4x,$$

for which $f'(x) = (3x-2)^2$, $f'(0) = 4$ and $f'(1) = 1$. Hence the greatest value is $\boxed{7/8}$.

16. E

At an intersection with x-coordinate x, put $p = x(x-2)$. The gradients of the two tangents are $x^2 - 1/2$ and $(x-2)^2 - 1/2$. Their product is

$$p^2 - \frac{1}{2}(2p+4) + \frac{1}{4} = \left(p - \frac{1}{2}\right)^2 - 2.$$

Perpendicularity therefore gives

$$\left(p - \frac{1}{2}\right)^2 = 1, \qquad p = -\frac{1}{2} \text{ or } p = \frac{3}{2}.$$

The difference of the two curve ordinates is

$$2x^2 - 4x + \frac{5}{3} - h.$$

Thus an intersection has $h = 2p + 5/3$, giving exactly two possible values of h. In each case the intersections are distinct, with horizontal separation

$$d = 2\sqrt{1+p}.$$

Consequently $d^2 = 2$ or $d^2 = 10$. The same value of p occurs at both intersections, so both tangent pairs are perpendicular.

If the intersection abscissae are r and s, with $r < s$, the positive vertical separation between the curves for $r < x < s$ is $2(x-r)(s-x)$. Expanding and integrating gives

$$A = \int_r^s 2(x-r)(s-x)\,\mathrm{d}x = \frac{(s-r)^3}{3} = \frac{d^3}{3}.$$

The two positive areas are distinct. Their squared sum is

$$\frac{2^3 + 10^3}{9} = \boxed{112}.$$

17. B

Keep track of the original counters contributing to each current counter. Every sequence of four pair choices has probability

$$\frac{1}{15 \cdot 10 \cdot 6 \cdot 3},$$

so such sequences can be counted equally.

The total is 22. The only unordered divisions of the original counters into two groups each totalling 11 are

$$\{4,7\} \mid \{1,2,3,5\}, \qquad \{1,3,7\} \mid \{2,4,5\}.$$

To check completeness, the group containing 7 must obtain its other total 4 either from 4 alone or from $1+3$.

A group of two counters takes one merge. A group of three has 3 possible internal merge sequences, and a group of four has $6 \cdot 3 = 18$.

For the first division, choose which of the four stages merges 4 with 7, then merge the other group: there are $4 \cdot 18 = 72$ sequences. For the second, choose the two stages used by one group and then its internal merges: there are

$$\binom{4}{2} \cdot 3 \cdot 3 = 54$$

sequences. Thus the conditioning event contains 126 sequences.

If the first pair is $1, 2$, only the first division is possible. Of the remaining three stages, choose the stage that merges $4, 7$; the other three current counters $\{1,2\}, 3, 5$ can be merged in 3 ways. This gives $3 \cdot 3 = 9$ sequences.

The required probability is $\boxed{9/126 = 1/14}$.

18. C

There are three cards in each remainder class on division by 3. The $\binom{9}{3} = 84$ initial hands have three types. In the descriptions below, a, b, c are the three distinct remainders.

Three equal remainders: a, a, a**.** There are 3 hands. Discarding one cannot win, as no a remains undrawn. If two are discarded, the replacements must be one b and one c. There are $3 \cdot 3 = 9$ such pairs among $\binom{6}{2} = 15$, giving optimum 3/5.

Exactly two equal remainders: a, a, b**.** There are

$$3 \cdot 2 \cdot \binom{3}{2} \cdot 3 = 54$$

hands. The six undrawn cards have remainders a, b, b, c, c, c. Discarding one a requires a replacement c, and wins with probability $3/6 = 1/2$; discarding b wins with probability $1/6$.

If two cards are discarded, retaining an a requires a pair with sum having remainder $2a$: only a b, c pair is available, giving $6/15$. Retaining b requires either a b, b pair or an a, c pair, giving $(1+3)/15$. Neither improves on $1/2$.

Three different remainders: a, b, c**.** There are $3^3 = 27$ hands. One discard wins with probability $2/6 = 1/3$. If an a is retained and two discarded, the undrawn cards have two of each remainder. The successful pairs are a, a or b, c, giving $(1 + 4)/15 = 1/3$.

These comparisons establish and attain the optimum for each hand. The maximum overall probability is

$$\frac{3 \cdot \frac{3}{5} + 54 \cdot \frac{1}{2} + 27 \cdot \frac{1}{3}}{84} = \boxed{\frac{9}{20}}.$$

19. F

All ordered draws are equally likely. A linear polynomial gives an arithmetic progression of five distinct integers. A positive common difference is 1 or 2: there are respectively 5 or 1 choices for the first term. Reversing gives 12 linear lists.

For a genuinely quadratic list, put

$$e = f(2) - 2f(1) + f(0), \qquad v = f(3) - f(1), \qquad m = f(2).$$

The five entries are

$$m + 2e - v, \quad m + \frac{e - v}{2}, \quad m, \quad m + \frac{e + v}{2}, \quad m + 2e + v.$$

Here e, v are integers of the same parity. Also $e \neq 0$; and $v \neq 0$, since the entries are distinct.

Replacing every entry y by $10 - y$ and reversing the list allow us to count first with $e > 0$ and $v > 0$. The last entry exceeds the middle one by $2e + v$, so $2e + v \leq 8$. The only candidate pairs (e, v) are

$$(1,1),\ (1,3),\ (1,5),\ (2,2),\ (2,4),\ (3,1).$$

The first, second, fourth and fifth pairs give repeated entries. The pair $(1, 5)$ gives a range of 10. The only valid pair is $(3, 1)$, giving offsets

$$5, \quad 1, \quad 0, \quad 2, \quad 7$$

from m. These fit among $1, \ldots, 9$ exactly when $m = 1$ or $m = 2$.

Each of these two lists gives four distinct lists by reflection of the values and reversal. Their signs of e and v are all different, so none is counted twice. Thus there are 8 genuinely quadratic lists.

The required conditional probability is $\boxed{8/(12 + 8) = 2/5}$.

20. A

The exact integral is $1/4 + a/6$. Since $f_a(0) = 0$ and $f_a(1) = 1$, the trapezium estimate is

$$T = \frac{1}{2}(f_a(t) + 1 - t).$$

Put $u = t - 1/2$ and $k = a - 3/2$. Expanding $T - (1/4 + a/6) = 0$ gives

$$12u^3 - 12ku^2 - 3u - k = 0, \qquad k = \frac{3u(4u^2 - 1)}{12u^2 + 1}, \qquad -\frac{1}{2} < u < \frac{1}{2}.$$

The expression for k is odd. To locate a candidate positive bound, consider the cubic

$$P_k(u) = 12u^3 - 12ku^2 - 3u - k.$$

For $k > 0$, its values at $u = \pm 1/2$ are both $-4k < 0$, and increasing k lowers $P_k(u)$ at every u. If its interior maximum is positive, a sufficiently small increase in k still leaves a positive interior value and negative endpoint values, so an interior root remains. This suggests a boundary where an interior maximum is zero. We first find this stationary-zero candidate, then prove the resulting bounds algebraically.

Differentiating only the cubic gives

$$P_k'(u) = 36u^2 - 24ku - 3.$$

Thus $P_k'(u) = 0$ requires $u \neq 0$ and gives $k = (12u^2 - 1)/(8u)$. Substitution into $P_k(u) = 0$ gives

$$48u^4 + 24u^2 - 1 = 0.$$

Writing $v = u^2 \geq 0$, the only non-negative root is

$$v_0 = \frac{2\sqrt{3} - 3}{12}.$$

Using $48v_0^2 + 24v_0 - 1 = 0$, the corresponding candidate satisfies

$$k^2 = \frac{(12v_0 - 1)^2}{64v_0} = 3v_0 = \frac{2\sqrt{3} - 3}{4}.$$

We now verify the bound for every allowed mesh. Set

$$K = 3v_0 = \frac{2\sqrt{3} - 3}{4}, \qquad v_1 = v_0 + \frac{1}{2} = \frac{2\sqrt{3} + 3}{12}.$$

Direct expansion gives

$$K(12v + 1)^2 - 9v(4v - 1)^2 = 144(v_1 - v)(v - v_0)^2.$$

For $0 \leq v < 1/4$, the right-hand side is non-negative because $v_1 > 1/4$. Therefore $k^2 \leq K$.

Equality is possible: $0 < v_0 < 1/4$, and the two choices $u = \sqrt{v_0}$ and $u = -\sqrt{v_0}$ give opposite values of k. Thus both $-\sqrt{K}$ and $\sqrt{K}$ are attained, with valid interior strip boundaries. The smallest and largest admissible values of a are consequently

$$m = \frac{3}{2} - \sqrt{K}, \qquad M = \frac{3}{2} + \sqrt{K}.$$

Since $0 < K < 1/4$, both are positive, as required by $a \geq 0$. Hence

$$mM = \frac{9}{4} - K = \boxed{3 - \frac{\sqrt{3}}{2}}.$$