**1.** The points $A = (-1, 0)$ and $B = (1, 0)$ are fixed. The line $L$ has equation

$$y = mx + c,$$

where $m$ and $c$ are real constants.

Consider the statements

S: $c^2 \leq m^2$.

T: $L$ meets every circle through $A$ and $B$ at two distinct points.

Is S necessary and/or sufficient for T?

**A** S is neither necessary nor sufficient for T

**B** S is necessary but not sufficient for T

**C** S is sufficient but not necessary for T

**D** S is necessary and sufficient for T

**2.** The quadratic polynomial

$$q(x) = ax^2 + bx + c$$

has integer coefficients, with $a > 0$. It satisfies $q(n) > 0$ for every integer $n$, but $q(x) < 0$ for at least one real number $x$.

A student writes

$$q(x) = a(x - t)^2 + h,$$

where $h < 0$, and chooses an integer $n$ nearest to $t$, so that $|n - t| \leq 1/2$.

Which of the following arguments validly rules out every integer $a$ with $1 \leq a \leq 4$?

**A** Since $q(n) > 0$, we have $h = q(n) - a(n - t)^2 > -a/4$. If $a \leq 4$, this implies $h \geq 0$, a contradiction.

**B** Because the coefficients are integers, $h$ is an integer. Thus $h \leq -1$, so if $a \leq 4$ then $q(n) \leq a/4 - 1 \leq 0$, a contradiction.

**C** For an integer nearest to $t$, we always have $|n - t| = 1/2$. Thus $1 \leq q(n) = a/4 + h < a/4$, so $a > 4$.

**D** The number $q(n)$ is a positive integer, so $q(n) \geq 1$. Hence $h = q(n) - a(n - t)^2 \geq 1 - a/4$. If $a \leq 4$, then $h \geq 0$, a contradiction.

**E** The inequalities $q(n) \geq 1$ and $(n - t)^2 \leq 1/4$ give $h \leq 1 - a/4$. Since $h < 0$, this forces $1 - a/4 < 0$, so $a > 4$.

**F** Since $(n - t)^2 \leq 1$ and $q(n) \geq 1$, we have $h \geq 1 - a$. If $a \leq 4$, then $h \geq -3$, contradicting $h < 0$.

**3.** Let $k \geq 0$. Consider the statements

S: $0 \leq k < 7 + 4\sqrt{3}$.

T: For every triangle of positive area whose side lengths $a$, $b$ and $c$ satisfy

$$a > 1, \qquad b > 1, \qquad c > 1,$$

the three numbers

$$a + \frac{k}{a}, \qquad b + \frac{k}{b}, \qquad c + \frac{k}{c}$$

can also be used as the side lengths of a triangle of positive area.

Is S necessary and/or sufficient for T?

**A** S is neither necessary nor sufficient for T

**B** S is necessary but not sufficient for T

**C** S is sufficient but not necessary for T

**D** S is necessary and sufficient for T

**4.** The vertices of a convex quadrilateral are $A, B, C, D$, in that order. The areas of the four triangles

$$ABC, \quad ABD, \quad ACD, \quad BCD$$

are 1, 2, 3 and 4, in some order.

The diagonals $AC$ and $BD$ meet at $O$. What is the area of the smallest of the four triangles $AOB$, $BOC$, $COD$ and $DOA$?

A quadrilateral is convex if each of its interior angles is less than 180°.

**A** $\frac{1}{5}$

**B** $\frac{2}{5}$

**C** $\frac{1}{2}$

**D** $\frac{3}{5}$

**E** $\frac{4}{5}$

**F** 1

**5.** The points $A = (5/3, 0)$ and $B = (0, 5/3)$ are fixed. A point $P = (x, y)$ lies on the circle

$$x^2 + y^2 = 1,$$

with $x \neq 0$.

The gradients of $AP$ and $BP$ are $m$ and $n$, respectively. What is the least possible value of $m/n$?

**A** $-\dfrac{9}{50}$

**B** $-\dfrac{9}{41}$

**C** $-\dfrac{1}{5}$

**D** $-\dfrac{4}{9}$

**E** $-\dfrac{1}{2}$

**F** $-\dfrac{1}{4}$

**6.** Three coins $A$, $B$ and $C$ are tossed independently. Their probabilities of landing heads are $p$, $q$ and $r$, respectively, where

$$0 < p < 1, \qquad 0 < q < 1, \qquad 0 < r < 1.$$

Let $S$ be the event that $A$ and $B$ show the same result, and let $T$ be the event that $B$ and $C$ show the same result.

Consider the propositions

P: Coin $B$ is fair, so $q = 1/2$.

Q: The events $S$ and $T$ are independent.

Is P necessary and/or sufficient for Q?

**A** P is neither necessary nor sufficient for Q

**B** P is necessary but not sufficient for Q

**C** P is sufficient but not necessary for Q

**D** P is necessary and sufficient for Q

**7.** Initially the six numbers $1, 2, 3, 4, 5, 6$ are written on a board.

In each move, one of the current entries is chosen uniformly at random and erased. Every other entry is then replaced by its arithmetic mean with the erased entry. This is repeated until only one entry remains.

What is the probability that the final entry is exactly 2?

**A** $\frac{1}{360}$

**B** $\frac{1}{180}$

**C** $\frac{1}{120}$

**D** $\frac{1}{60}$

**E** $\frac{1}{30}$

**F** $\frac{1}{15}$

**8.** Eight distinct integers are selected from

$$1, 2, 3, \ldots, 16.$$

No selected integer is a factor of any other selected integer.

What is the least possible sum of the eight selected integers?

**A** 64

**B** 70

**C** 72

**D** 75

**E** 79

**F** 100

**9.** A real-valued function $f$, defined for every real number, satisfies

$$f(x+1) = f(x) + 2x + 1, \qquad f(2x) = 4f(x)$$

for every real $x$. Also, $0 \le f(x) \le 1$ whenever $0 \le x \le 1$.

A student wishes to prove that $f(x) = x^2$ for every real $x$. The student puts $g(x) = f(x) - x^2$ and correctly obtains

$$g(x+1) = g(x), \qquad g(2x) = 4g(x), \qquad |g(x)| \le 1 \text{ when } 0 \le x \le 1.$$

For a fixed real $x$ and each positive integer $n$, the student chooses an integer $k_n$ such that

$$0 \le 2^n x - k_n < 1.$$

Which of the following validly completes the proof, without assuming that $f$ or $g$ is continuous?

**A** Since $g$ has period 1, $g(2^n x) = g(x)$. Combining this with $g(2^n x) = 4^n g(x)$ gives $g(x) = 0$.

**B** The bound on $f$ gives $|4^n f(x) - k_n^2| \le 1$. Dividing by $4^n$ and letting $n$ increase gives $f(x) = x^2$.

**C** One integer $k$ can be chosen so that $0 \le 2^n x - k < 1$ for every $n$. Thus $g$ is bounded on the sequence $2^n x$, forcing $g(x) = 0$.

**D** Repeated scaling gives $g(x) = 4^{-n} g(x/2^n)$. Since $x/2^n$ approaches 0, the bound on $g$ gives $g(x) = 0$.

**E** Integer translations give $g(2^n x - k_n) = g(2^n x) = 4^n g(x)$. Hence $|g(x)| \le 4^{-n}$ for every positive integer $n$, so $g(x) = 0$.

**F** The numbers $2^n x - k_n$ lie between 0 and 1, so they approach 0 as $n$ increases. Therefore $g(2^n x - k_n)$ approaches $g(0) = 0$, forcing $g(x) = 0$.

**10.** An array has 5 labelled rows and 4 labelled columns. Each of its 20 cells is coloured red or blue.

Whenever two distinct rows and two distinct columns are chosen, the four cells at their intersections must not all be the same colour.

A proof that there are exactly 720 such colourings has been split into five blocks. Two colourings are different if at least one labelled cell has a different colour.

I. Both necessity and sufficiency have now been established. Assign five different patterns to the five labelled rows in $6 \times 5 \times 4 \times 3 \times 2 = 720$ ways.

II. An all-blue row forces every other row to have at least three red cells, so two of those rows share two red columns. A row with one red cell forces each of the other four rows to have at least two red cells in its three blue columns. Among the three pairs of those columns, two of the four rows must share a red pair. Both cases give a forbidden rectangle. Interchanging colours also rules out all-red rows and rows with three red cells. Thus every row has two cells of each colour.

III. This last necessary condition is also sufficient: two different balanced row patterns cannot share both red positions or both blue positions, so choosing different patterns creates no monochromatic rectangle.

IV. Each row must therefore be one of the six patterns with exactly two red positions among four labelled columns.

V. Of these six patterns, no one can appear in two rows, because their two red positions would give a forbidden rectangle. Hence a valid colouring must use five distinct patterns.

Which ordering gives a complete proof, with every reference to an earlier conclusion justified when it is used?

**A** II, IV, III, V, I

**B** IV, II, V, III, I

**C** II, IV, V, III, I

**D** II, V, IV, III, I

**E** II, IV, V, I, III

**F** V, II, IV, III, I

**11.** A polynomial $P(x)$ has degree 4 and coefficient 1 for $x^4$. Two of its real roots are $a$ and $b$, where $b - a = 2$.

Consider the statements

S: The tangents to $y = P(x)$ at $(a, 0)$ and $(b, 0)$ have the same gradient.

T: $\displaystyle\int_a^b P(x)\,\mathrm{d}x = \frac{16}{15}$.

Which of the following correctly describes the relationship between S and T?

**A** S is neither necessary nor sufficient for T

**B** S is necessary but not sufficient for T

**C** S is sufficient but not necessary for T

**D** S is necessary and sufficient for T

**12.** Each use of a machine produces one of the numbers 1, 2 and 3. Each number has a fixed positive probability of being produced, and the results of different uses are independent.

If $X$ and $Y$ are the results of two uses, it is known that

$$\Pr(X < Y) = \frac{1}{4}, \qquad \Pr(X + Y \text{ is odd}) = \frac{3}{8}.$$

The machine is used three more times, producing $U$, $V$ and $W$.

Given that $U + V + W$ is even, what is the probability that $U$, $V$ and $W$ are all different?

**A** $\frac{3}{32}$

**B** $\frac{3}{28}$

**C** $\frac{3}{16}$

**D** $\frac{1}{4}$

**E** $\frac{3}{7}$

**F** $\frac{3}{14}$

**13.** For a real constant $k$ with $0 \leq k \leq 2$, define

$$a_n = n + \frac{(-1)^n k}{n} \qquad (n = 1, 2, 3, \ldots).$$

Consider the statement:

For every real number $\varepsilon > 0$, there exist distinct positive integers $m$ and $n$ such that

$$|a_m - a_n| < \varepsilon.$$

For which values of $k$ is this statement true?

**A** $k = \frac{6}{5}$ only

**B** $k = 1$ or $k = \frac{6}{5}$

**C** $1 \leq k \leq 2$

**D** $\frac{6}{5} \leq k \leq 2$

**E** Every $k$ with $0 \leq k \leq 2$

**F** There are no such values of $k$

**14.** Triangle $ABC$ is right-angled at $A$, with $AB = AC = 5$.

Three circles have centres $A$, $B$ and $C$ and radii $r$, $2r$ and $\sqrt{2}\,r$, respectively, where $r > 0$.

Every point inside or on the boundary of triangle $ABC$ is required to lie inside or on at least one of the three circles.

What is the least possible value of $r^2$?

**A** $\dfrac{25}{9}$

**B** $25(3 - 2\sqrt{2})$

**C** $5$

**D** $\dfrac{25}{4}$

**E** $\dfrac{25}{2}$

**F** $25$

**15.** Let

$$f(x) = x^4 - \frac{3}{2}x^2,$$

and let $k$ be a real constant. Consider the statements

S: For every real number $h > 0$, the equation

$$f(x+h) - f(x) = kh$$

has exactly one real solution for $x$.

T: $k < -1$ or $k > 1$.

Is S necessary and/or sufficient for T?

**A** S is neither necessary nor sufficient for T

**B** S is necessary but not sufficient for T

**C** S is sufficient but not necessary for T

**D** S is necessary and sufficient for T

**16.** Nine players have distinct ranks $1, 2, \ldots, 9$, with rank 1 the strongest. Whenever two players play, the player with the smaller rank wins.

At each stage, one pair is chosen uniformly at random from all pairs of players still present. They play one match, and the loser leaves. A new pair is then chosen from the remaining players. This continues until only one player remains.

What is the probability that the player of rank 3 plays in the final match?

**A** $\frac{1}{8}$

**B** $\frac{1}{6}$

**C** $\frac{5}{24}$

**D** $\frac{1}{4}$

**E** $\frac{1}{3}$

**F** $\frac{5}{12}$

**17.** A point $P$ lies on the parabola $y = x^2$. Its reflections in the three lines

$$x = 0, \qquad y = 0, \qquad 3x + 2y = 12$$

are three distinct collinear points.

What is the sum of the reciprocals of the $x$-coordinates of all possible points $P$?

**A** $-\frac{5}{4}$

**B** $-1$

**C** $-\frac{3}{4}$

**D** $-\frac{1}{4}$

**E** $\frac{1}{4}$

**F** $\frac{5}{4}$

**18.** For a real constant $k$, define

$$f_k(x) = x^3 - kx^2 + 10x.$$

Consider the statements

S: $k \geq 13/2$.

T: For every pair of distinct positive real numbers $a, b$ satisfying

$$ab = 1 \qquad \text{and} \qquad f_k(a) = f_k(b),$$

the tangents to $y = f_k(x)$ at $x = a$ and $x = b$ both have strictly positive gradient.

Is S necessary and/or sufficient for T?

**A** S is neither necessary nor sufficient for T

**B** S is necessary but not sufficient for T

**C** S is sufficient but not necessary for T

**D** S is necessary and sufficient for T

**19.** The eight distinct integers

$$1, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 12, \quad 24$$

are placed at the eight vertices of a fixed cube, with every placement equally likely.

It is given that, for each of the six faces, the product of the four integers at its vertices is a perfect square.

What is the probability that the integers $1, 2, 3, 6$ occupy the four vertices of one face?

**A** $\dfrac{3}{35}$

**B** $\dfrac{3}{28}$

**C** $\dfrac{1}{7}$

**D** $\dfrac{3}{14}$

**E** $\dfrac{1}{4}$

**F** $\dfrac{3}{7}$

**20.** The four numbers

$$\sin\alpha, \qquad \cos\alpha, \qquad \sin\beta, \qquad \cos\beta,$$

when arranged in increasing order, form an arithmetic progression with positive common difference. The angles $\alpha$ and $\beta$ are real, and $\cos\alpha\cos\beta \neq 0$.

What is the sum of all the distinct possible values of

$$\tan\alpha\tan\beta\,?$$

**A** 1

**B** $\dfrac{10}{9}$

**C** $\dfrac{82}{9}$

**D** 10

**E** $\dfrac{91}{9}$

**F** $\dfrac{100}{9}$