1. B

Every circle through A and B has its centre on the y-axis. Its equation is therefore

$$x^2 + y^2 - 2ty = 1$$

for some real t, and every real t gives such a circle.

Substitute $y = mx + c$. The intersection points correspond to roots of

$$(1 + m^2)x^2 + 2m(c - t)x + c^2 - 2ct - 1 = 0.$$

The coefficient of x^2 is positive. One quarter of its discriminant is

$$D(t) = m^2t^2 + 2ct + 1 + m^2 - c^2.$$

Thus the required condition is $D(t) > 0$ for every real t.

If $m \neq 0$, completing the square gives

$$D(t) = m^2 \left(t + \frac{c}{m^2} \right)^2 + \frac{(1 + m^2)(m^2 - c^2)}{m^2}.$$

Its minimum is strictly positive precisely when $c^2 < m^2$. Equality $c^2 = m^2$ does not suffice: one of the circles is then tangent to L.

If $m = 0$, then $D(t) = 2ct + 1 - c^2$. This is positive for every real t precisely when $c = 0$, in which case L is the line AB and meets each circle at A and B.

Combining the cases gives $\boxed{c^2 < m^2, \text{ or } m = c = 0}$.

Therefore T implies S: either of the two cases in the boxed condition gives $c^2 \leq m^2$.

The converse fails. Take $m = c = 1$, so S is true. The circle

$$x^2 + y^2 + 2y = 1$$

passes through A and B. Substituting $y = x + 1$ gives $2(x + 1)^2 = 0$, so the line meets this circle at just the point A, not at two distinct points. Thus T is false.

Hence S is necessary but not sufficient for T, giving B.

2. D

Write

$$q(x) = a(x-t)^2 + h,$$

where t is the x-coordinate of the vertex and h is the minimum value. The existence of a negative value means $h < 0$.

Choose an integer n nearest to t. Then $|n-t| \le 1/2$. Because q has integer coefficients, $q(n)$ is a positive integer, and hence $q(n) \ge 1$. It follows that

$$h = q(n) - a(n-t)^2 \ge 1 - \frac{a}{4}.$$

If $a \le 4$, this would give $h \ge 0$, a contradiction. Therefore the positive integer a must be at least 5.

This bound is attained by

$$q(x) = 5x^2 - 5x + 1 = 5x(x-1) + 1.$$

For every integer n, either $n \le 0$ or $n \ge 1$, so $n(n-1) \ge 0$ and $q(n) \ge 1$. But $q(1/2) = -1/4 < 0$.

The smallest possible value is $\boxed{5}$.

The missing proof fragment is D: it uses both the integrality of $q(n)$ and the correct direction of the nearest-integer bound.

A only proves $h > -a/4$; a negative h can satisfy this, so it cannot yield the claimed contradiction. B incorrectly treats the minimum h as an integer: the attaining example above has $h = -1/4$.

C replaces $|n-t| \le 1/2$ by an unjustified equality. For example,

$$q(x) = 6x^2 - 5x + 1 = 6(x - 5/12)^2 - 1/24$$

has all the required properties, but the nearest integer to $t = 5/12$ is 0 and $|n-t| = 5/12$. Indeed, it is positive at every integer: for $n \le 0$ each nonconstant term is non-negative, and for $n \ge 1$ we have $6n^2 - 5n = n(6n-5) \ge 1$. Its minimum is negative.

E reverses the bound on h. Even an upper bound on a negative quantity need not be negative. F derives a valid but weaker bound: $h \ge -3$ is compatible with $h < 0$. Therefore no option other than D supplies a valid argument as written.

3. C

Put

$$K = 7 + 4\sqrt{3} = (2+\sqrt{3})^2, \qquad F(t) = t + \frac{k}{t}.$$

All three transformed lengths are positive.

First suppose $0 \le k \le K$. Label the original sides so that $1 < a \le b \le c$. The original triangle inequalities give

$$F(a) + F(b) - F(c) = (a+b-c) + k\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) > 0,$$
$$F(a) + F(c) - F(b) = (a+c-b) + k\left(\frac{1}{a} + \frac{1}{c} - \frac{1}{b}\right) > 0.$$

In each line the first bracket is positive and the reciprocal bracket is positive, because $a \le b \le c$.

It remains to show that $F(b) + F(c) > F(a)$: the shortest original side might become the longest transformed side. From $(b-c)^2 \ge 0$,

$$\frac{1}{b} + \frac{1}{c} - \frac{4}{b+c} = \frac{(b-c)^2}{bc(b+c)} \ge 0.$$

Write $u = (b+c)/a$, so $u \ge 2$. Then

$$\begin{aligned} F(b) + F(c) - F(a) &\ge b + c - a + k\left(\frac{4}{b+c} - \frac{1}{a}\right) \\ &= a\left(u - 1 + \frac{k}{a^2}\left(\frac{4}{u} - 1\right)\right). \end{aligned}$$

If $2 \le u \le 4$, the bracket is positive. If $u > 4$, then $a > 1$ and $k \le K$ give $k/a^2 < K$. Since $4/u - 1 < 0$, the displayed lower bound is strictly greater than

$$\frac{a}{u}\left(u^2 - (K+1)u + 4K\right) = \frac{a}{u}\left(u - 4 - 2\sqrt{3}\right)^2 \ge 0.$$

Thus all three triangle inequalities are strict, proving sufficiency, including $k = K$.

Conversely, suppose $k > K$ and put $r = \sqrt{k}$. Choose

$$a = (2-\sqrt{3})r, \qquad b = c = r.$$

Since $r > 2 + \sqrt{3}$ and $(2-\sqrt{3})(2+\sqrt{3}) = 1$, all three lengths exceed 1. Also $0 < a < 2r$, so they form a triangle of positive area. But

$$F(a) = \left((2-\sqrt{3}) + (2+\sqrt{3})\right) r = 4r, \qquad F(b) = F(c) = 2r.$$

The transformed lengths therefore give a degenerate triangle, so the statement is false.

The necessary and sufficient condition is

$$\boxed{0 \le k \le 7 + 4\sqrt{3}}.$$

The upper endpoint is included because the equality construction there would have $a = 1$, which is excluded by the original strict inequality.

S restricts k to $0 \le k < K$, so S implies T. However, at $k = K = 7 + 4\sqrt{3}$, T is true by the strict inequalities proved above, while S is false. Therefore S is sufficient but not necessary for T, giving C.

4. B

Each diagonal divides the quadrilateral into two triangles. Thus the four listed areas split into two pairs having the same sum, equal to the quadrilateral's area. The only possible pairing is

$$1 + 4 = 2 + 3 = 5.$$

The quadrilateral therefore has area 5.

Temporarily relabel its vertices, reversing their order if necessary, so that

$$[ABC] = 1, \qquad [ABD] = 2, \qquad [ACD] = 4, \qquad [BCD] = 3.$$

This is possible: first label the triangle of area 1 as ABC; its opposite triangle across AC has area 4, and interchanging A and C if needed places the triangle of area 2 at ABD. Relabelling does not change the four small areas being sought.

Put

$$[AOB] = x, \quad [BOC] = y, \quad [COD] = z, \quad [DOA] = w.$$

Adding the appropriate small triangles gives

$$x + y = 1, \qquad x + w = 2, \qquad z + w = 4, \qquad y + z = 3.$$

Also, triangles AOB and BOC have bases AO and OC on the same line and a common height, as do triangles AOD and COD. Hence

$$\frac{x}{y} = \frac{AO}{OC} = \frac{w}{z}, \qquad xz = yw.$$

Substitute $y = 1 - x$, $w = 2 - x$ and $z = 2 + x$:

$$x(2 + x) = (1 - x)(2 - x), \qquad 5x = 2.$$

The four small areas are therefore

$$\frac{2}{5}, \quad \frac{3}{5}, \quad \frac{12}{5}, \quad \frac{8}{5},$$

whose smallest is $\boxed{2/5}$.

The premises are attainable. For example, take $A = (1, 0)$, $B = (0, 4/5)$, $C = (-3/2, 0)$ and $D = (0, -16/5)$. These vertices form a convex quadrilateral, and direct base–height calculations give exactly the four small areas displayed above and the required large areas.

5. F

Because $-1 \le x, y \le 1$ and $x \neq 0$, both gradients are finite and $n \neq 0$. They give

$$\frac{m}{n} = \frac{y/(x-5/3)}{(y-5/3)/x} = \frac{xy}{(x-5/3)(y-5/3)}.$$

Put $s = x + y$. The circle equation implies $2xy = s^2 - 1$, so

$$\frac{m}{n} = \frac{9(s^2-1)}{9s^2 - 30s + 41}.$$

The denominator is $(3s-5)^2 + 16 > 0$. Completing a square gives

$$\frac{9(s^2-1)}{9s^2-30s+41} + \frac{1}{4} = \frac{5(3s-1)^2}{4(9s^2-30s+41)} \ge 0.$$

Thus $m/n \ge -1/4$ for every permitted point, with equality exactly when $s = 1/3$.

This value of s is attainable. Take

$$x = \frac{1+\sqrt{17}}{6}, \qquad y = \frac{1-\sqrt{17}}{6}.$$

Then $x + y = 1/3$, $x \neq 0$, and

$$x^2 + y^2 = \frac{(1+\sqrt{17})^2 + (1-\sqrt{17})^2}{36} = 1.$$

The point is therefore admissible and attains the bound. The least possible ratio is $\boxed{-1/4}$.

In particular, imposing the tempting extra symmetry $x = -y$ would instead give $-9/41$, which is not the minimum.

6. A

Condition on the result of coin B. If B is heads, then S requires A to be heads and T requires C to be heads. If B is tails, both require the corresponding outer coin to be tails. Independence of the three tosses therefore gives

$$\begin{aligned}\Pr(S) &= pq + (1-p)(1-q),\\ \Pr(T) &= qr + (1-q)(1-r),\\ \Pr(S \text{ and } T) &= pqr + (1-p)(1-q)(1-r).\end{aligned}$$

Expanding and subtracting yields

$$\Pr(S \text{ and } T) - \Pr(S)\Pr(T) = q(1-q)(2p-1)(2r-1).$$

The two events are independent precisely when this difference is zero. Since $0 < q < 1$, the factor $q(1-q)$ is positive, so the difference is zero exactly when

$$p = \frac{1}{2} \qquad \text{or} \qquad r = \frac{1}{2}.$$

Either outer coin being fair is sufficient, and at least one of them must be fair. The middle coin need not be fair.

Thus Q is true exactly when $p = 1/2$ or $r = 1/2$.

P does not imply Q. Take

$$p = r = \frac{3}{4}, \qquad q = \frac{1}{2}.$$

Then P is true, but the independence difference is

$$\frac{1}{2}\left(1 - \frac{1}{2}\right)\left(2 \cdot \frac{3}{4} - 1\right)^2 = \frac{1}{16} \neq 0,$$

so Q is false.

Q does not imply P either. Take

$$p = \frac{1}{2}, \qquad q = \frac{1}{3}, \qquad r = \frac{3}{4}.$$

The independence difference is zero because $2p - 1 = 0$, but the middle coin is not fair. Thus Q is true and P is false.

Consequently P is neither necessary nor sufficient for Q, giving A. Here S, T denote events throughout, and P,Q denote propositions.

7. A

Keep a label on each entry recording its original number. Write $a_1, a_2, \ldots, a_5$ for the labels erased in order and a_6 for the surviving label. These form a permutation of $1, 2, 3, 4, 5, 6$.

After the first move, the entry with label b has value $(a_1 + b)/2$. After the next move it has value $a_1/2 + a_2/4 + b/4$. Continuing the same calculation shows that the final value is

$$\frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} + \frac{a_5 + a_6}{32}.$$

As $a_1 + \cdots + a_6 = 21$, the condition that this equal 2 becomes

$$15a_1 + 7a_2 + 3a_3 + a_4 = 43. \qquad (*)$$

For positive unequal coefficients, the least weighted sum of a chosen collection of numbers pairs the largest coefficient with the smallest number: assigning $u < v$ to weights $U < V$ in the same order, rather than in opposite orders, increases the sum by $(V - U)(v - u) > 0$.

If $a_1 \geq 2$, the left side of $(*)$ is at least

$$15 \cdot 2 + 7 \cdot 1 + 3 \cdot 3 + 4 = 50 > 43.$$

Thus $a_1 = 1$. If now $a_2 \geq 3$, its least possible value is at least

$$15 + 7 \cdot 3 + 3 \cdot 2 + 4 = 46 > 43.$$

So $a_2 = 2$. If $a_3 \geq 4$, the left side is at least

$$15 + 14 + 3 \cdot 4 + 3 = 44 > 43.$$

Hence $a_3 = 3$, after which $(*)$ gives $a_4 = 5$. The remaining labels are 4 and 6, and either may be erased next; their order does not change the final value.

Every remaining label is equally likely at each move. The required first four labels have probability

$$\frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} = \boxed{\frac{1}{360}}.$$

There is no further restriction on the last move, so this is the required probability.

8. D

Partition the available integers into the eight lists

$$\begin{array}{llll} 1,2,4,8,16; & 3,6,12; & 5,10; & 7,14; \\ 9; & 11; & 13; & 15. \end{array}$$

Within each list, every entry is a factor of all the later entries. Thus at most one entry from each list can be selected. Since eight entries are selected from eight lists, exactly one must come from each.

In particular, $9, 11, 13, 15$ must be selected. The selected entry from $3, 6, 12$ cannot be 3, because 3 is a factor of the forced entry 9; it is therefore at least 6. The selected entry from $5, 10$ cannot be 5, because 15 is selected, so it must be 10.

The selected entry from $1, 2, 4, 8, 16$ can be neither 1 nor 2, since each divides the forced entry 10. It is at least 4. The entry from $7, 14$ is at least 7.

Consequently the sum is at least

$$4 + 6 + 10 + 7 + 9 + 11 + 13 + 15 = 75.$$

This bound is attained by selecting

$$4, 6, 7, 9, 10, 11, 13, 15.$$

No entry is a factor of any larger entry in this list: 4 has no selected multiple, 6 has no selected multiple, 7 has no selected multiple, and every entry from 9 onwards exceeds half the largest selected entry. The least possible sum is therefore $\boxed{75}$.

9. E

Option E uses only the established relations. From $g(t+1) = g(t)$, repeated forward or backward translation gives

$$g(t-j) = g(t)$$

for every real t and every integer j. Repeated use of $g(2t) = 4g(t)$ gives $g(2^n x) = 4^n g(x)$. Thus

$$4^n g(x) = g(2^n x - k_n).$$

The input on the right satisfies $0 \le 2^n x - k_n < 1$, so its absolute value is at most 1. Therefore

$$|g(x)| \le 4^{-n}$$

for every positive integer n. If $|g(x)|$ were positive, sufficiently large powers of 4 would violate this bound. Hence $g(x) = 0$. This holds for each fixed real x, proving $f(x) = x^2$. This function satisfies all the original conditions.

A incorrectly treats multiplication of the input by an integer as an integer translation. B omits a cross term: the correct translation formula is

$$f(2^n x - k_n) = 4^n f(x) - 2^{n+1} k_n x + k_n^2.$$

For example, even the valid function $f(x) = x^2$ violates B's asserted bound at $x = 3/4$, $n = 1$ and $k_n = 1$.

C changes the order of the choices: the integer may depend on n. For $x = 1$, no fixed integer k works for all n. D reverses the scaling factor; the given relation yields $g(x) = 4^n g(x/2^n)$, not $4^{-n} g(x/2^n)$. Its limiting argument would also require justification absent from the stated bound. F confuses bounded inputs with inputs approaching 0: when $x = 1/3$, the displayed inputs alternate between $2/3$ and $1/3$. Continuity has not been assumed either. Only E is a valid completion as written.

10. C

The correct order is II, IV, V, III, I, giving C.

Block II first establishes that every row is balanced. If one row is all blue, each other row must have at least three red cells; two of those rows then have at least two common red columns. This is forbidden. If one row has one red cell and three blue cells, every other row must have at least two red cells in those three blue columns. Choose one such pair from each of the four other rows. There are only three possible column pairs, so two rows share a red pair, again giving a forbidden rectangle. Interchanging the colours handles the other two unbalanced cases.

Block IV can now use this conclusion: a balanced row is determined by choosing its two red columns, with $\binom{4}{2} = 6$ possibilities.

Block V refers to these six patterns and proves the further necessary condition that the five rows use different patterns. Repeating a pattern would repeat both red positions in two rows and create a forbidden rectangle.

Block III refers to this last necessary condition, namely the use of different balanced patterns, and proves that it is sufficient. Two different balanced rows cannot share both red positions. They cannot share both blue positions either, since their complementary red positions would then also coincide. These are exactly the two ways that two rows could produce a monochromatic rectangle.

Only after both directions have been proved can block I conclude that the valid colourings are precisely the ordered choices of five different patterns from six. The row labels give

$$6 \times 5 \times 4 \times 3 \times 2 = 720.$$

The column labels are already included in the choice of the two red positions, so no further factor is needed.

A puts III before V, before the distinct-pattern necessary condition has been established. B starts with IV's unsupported "therefore". D and F use V's "these six patterns" before IV has identified them. E places I's claim that necessity and sufficiency have both been established before the sufficiency proof in III. Thus C is the only ordering in which all the stated references are justified.

11. D

Translate the horizontal coordinate so that the midpoint of the two roots is at 0. Since $b - a = 2$, write

$$t = x - \frac{a+b}{2}, \qquad Q(t) = P\left(t + \frac{a+b}{2}\right).$$

The translated polynomial is still monic of degree 4, and $Q(-1) = Q(1) = 0$. By the Factor Theorem it has the form

$$Q(t) = (t^2 - 1)(t^2 + ut + v)$$

for some real u, v.

Expanding and differentiating gives

$$Q'(t) = 4t^3 + 3ut^2 + 2(v-1)t - u.$$

Thus

$$Q'(1) - Q'(-1) = 4(1 + v).$$

The two gradients are equal if and only if $v = -1$. Therefore S is equivalent to $v = -1$.

On the other hand, the odd-power terms have integral 0 from -1 to 1. Hence

$$\begin{aligned}\int_a^b P(x)\,\mathrm{d}x &= \int_{-1}^{1} [t^4 + (v-1)t^2 - v]\,\mathrm{d}t \\ &= \frac{2}{5} + \frac{2(v-1)}{3} - 2v = -\frac{4}{15} - \frac{4v}{3}.\end{aligned}$$

This equals $16/15$ if and only if $v = -1$. T is therefore equivalent to exactly the same condition as S.

Both directions hold, so S is necessary and sufficient for T, giving D. The equivalence does not require any assumption about the other roots or the common gradient being nonzero. For example, $(x^2 - 1)^2$ makes both S and T true, whereas $(x^2 - 1)x^2$ makes both false.

12. F

Let p, q, r be the probabilities of producing $1, 2, 3$, respectively. Then $p + q + r = 1$, with all three positive.

The two results have the same distribution, so $\Pr(X > Y) = \Pr(X < Y) = 1/4$. Hence

$$p^2 + q^2 + r^2 = \Pr(X = Y) = \frac{1}{2}.$$

Exactly one of X, Y is 2 when their sum is odd. Therefore

$$2q(1 - q) = \frac{3}{8}, \qquad q = \frac{1}{4} \text{ or } q = \frac{3}{4}.$$

The second choice is impossible, since then $q^2 = 9/16 > 1/2$, already exceeding $p^2 + q^2 + r^2$. Thus $q = 1/4$ and $p + r = 3/4$. It follows that

$$2pr = (p + r)^2 - (p^2 + r^2) = \frac{9}{16} - \left(\frac{1}{2} - \frac{1}{16}\right) = \frac{1}{8},$$

so $pr = 1/16$.

All three results are different precisely when they are $1, 2, 3$ in one of six orders. This event has probability

$$6pqr = 6 \cdot \frac{1}{4} \cdot \frac{1}{16} = \frac{3}{32},$$

and its sum is automatically even.

For the sum of three results to be even, either all three results are 2, or exactly one is 2 and the other two are odd. These disjoint cases give

$$\Pr(U + V + W \text{ is even}) = q^3 + 3q(p + r)^2 = \frac{1}{64} + \frac{27}{64} = \frac{7}{16}.$$

The required conditional probability is therefore

$$\frac{3/32}{7/16} = \boxed{\frac{3}{14}}.$$

The initial data are attainable: $q = 1/4$ and $p, r = (3 \pm \sqrt{5})/8$, in either order, are positive probabilities with the required sum and product.

13. A

The first three terms are

$$a_1 = 1 - k, \qquad a_2 = 2 + \frac{k}{2}, \qquad a_3 = 3 - \frac{k}{3}.$$

In particular,

$$a_3 - a_2 = 1 - \frac{5k}{6}.$$

When $k = 6/5$, the distinct indices 2 and 3 give equal terms. Taking $m = 2$, $n = 3$ therefore works for every $\varepsilon > 0$, so the statement is true.

We prove that no other k works; having infinitely many terms does not by itself force arbitrarily small gaps. For consecutive terms with $n \geq 4$,

$$a_{n+1} - a_n = \begin{cases} 1 + k\left(\dfrac{1}{n} + \dfrac{1}{n+1}\right), & n \text{ odd}, \\ 1 - k\left(\dfrac{1}{n} + \dfrac{1}{n+1}\right), & n \text{ even}. \end{cases}$$

The first expression is at least 1. In the second, $n \geq 4$ and $k \leq 2$, so

$$a_{n+1} - a_n \geq 1 - 2\left(\frac{1}{4} + \frac{1}{5}\right) = \frac{1}{10}.$$

Thus the terms from a_4 onwards are strictly increasing, and any two of them differ by at least $1/10$.

Also $a_1 \leq 1$, while $a_2 \geq 2$ and $a_3 \geq 7/3$. Both a_2 and a_3 are at most 3, whereas $a_4 = 4 + k/4 \geq 4$. Hence every gap involving the first three terms is at least 1, except possibly the gap between a_2 and a_3.

If $k \neq 6/5$, set

$$\delta = \min\left\{\frac{1}{10}, \left|1 - \frac{5k}{6}\right|\right\} > 0.$$

All distinct-index terms then differ by at least δ. Taking $\varepsilon = \delta$ makes the stated strict inequality impossible. The statement is false for this k.

Consequently the complete answer is $\boxed{k = 6/5}$ only.

14. C

Place

$$A = (0,0), \qquad B = (5,0), \qquad C = (0,5).$$

A point (x, y) is in or on the triangle precisely when $x \geq 0$, $y \geq 0$ and $x + y \leq 5$.

To find a candidate for the sharp bound, put $t = r^2$ and seek a point on all three circle boundaries. Its coordinates would satisfy

$$x^2 + y^2 = t, \qquad (x-5)^2 + y^2 = 4t, \qquad x^2 + (y-5)^2 = 2t.$$

Subtracting the first equation from the other two gives

$$x = \frac{25 - 3t}{10}, \qquad y = \frac{25 - t}{10}.$$

Substitution into $x^2 + y^2 = t$ yields $(t-5)(t-25) = 0$. The value $t = 5$ gives the interior point $P = (1, 2)$; the other value gives $(-5, 0)$, outside the triangle. We now prove, rather than assume, that $t = 5$ is the required bound.

The squared distances from P to A, B and C are 5, 20 and 10, respectively. Thus if $r^2 < 5$, all three circles miss P. Consequently $r^2 \geq 5$ is necessary.

It remains to prove that $r^2 = 5$ covers the whole triangle. Suppose that a point (x, y) in the triangle lies outside all three circles, and put $s = x^2 + y^2$. The three strict inequalities would be

$$s - 5 > 0, \qquad s - 10x + 5 > 0, \qquad s - 10y + 15 > 0.$$

Multiply them by $5 - x - y$, x and y, respectively, and add. These multipliers are non-negative and have sum 5, so the resulting left-hand side must be positive. But direct expansion gives

$$\begin{aligned}(5 - x - y)(s-5) + x(s - 10x + 5) + y(s - 10y + 15) \\ = -5\big((x-1)^2 + (y-2)^2\big) \leq 0,\end{aligned}$$

a contradiction. Hence no point of the triangle is outside all three circles when $r^2 = 5$.

The lower bound is attained, so the least possible value is $\boxed{5}$.

15. B

For a fixed $h > 0$, put $u = x + h/2$. Expanding about this midpoint gives

$$\frac{f(x+h) - f(x)}{h} = 4u^3 + (h^2 - 3)u.$$

Thus the required number of solutions is the number of intersections of

$$y = g_h(u) = 4u^3 + (h^2 - 3)u$$

with the horizontal line $y = k$.

If $h \geq \sqrt{3}$, the function is strictly increasing, and its values extend without bound in both directions, so there is exactly one intersection. This includes $h = \sqrt{3}$, when $g_h(u) = 4u^3$.

If $0 < h < \sqrt{3}$, write

$$t = \sqrt{\frac{3 - h^2}{12}} > 0.$$

The derivative $g_h'(u) = 12u^2 + h^2 - 3$ vanishes at $u = \pm t$. The local maximum and minimum are

$$g_h(-t) = 8t^3 = M, \qquad g_h(t) = -8t^3 = -M, \qquad M = \left(1 - \frac{h^2}{3}\right)^{3/2}.$$

The graph has three strictly monotone portions. It therefore has one intersection with $y = k$ when $|k| > M$, two distinct intersections when $|k| = M$, and three when $|k| < M$.

For every permitted h in this case, $0 < M < 1$. Consequently $|k| \geq 1$ is sufficient for exactly one solution for every $h > 0$. In particular, $k = \pm 1$ are included: the values $M = 1$ would require the excluded value $h = 0$.

Conversely, suppose $q = |k| < 1$. Choose

$$h = \sqrt{3\left(1 - \left(\frac{1+q}{2}\right)^{2/3}\right)}.$$

Then $0 < h < \sqrt{3}$ and $M = (1 + q)/2 > q$. This choice gives three real solutions, so the property fails.

The necessary and sufficient condition is therefore $\boxed{k \leq -1 \text{ or } k \geq 1}$.

This boxed condition is equivalent to S. T is the stricter condition $|k| > 1$, so T implies S.

Conversely, take $k = 1$. For every $h > 0$, the preceding proof gives exactly one real solution, because $M < 1$ whenever the cubic has two turning points; thus S is true. But T is false at $k = 1$.

Therefore S is necessary but not sufficient for T, giving B. The proposition order matters: S denotes the for-every-h uniqueness property, whereas T denotes $|k| > 1$.

16. C

The player of rank 1 is never eliminated. First find the probability p_2 that the player of rank 2 reaches the final match.

Whenever k players remain and rank 2 is still present, the only pair whose selection can eliminate rank 2 is the pair of ranks 1 and 2. There are $k(k-1)/2$ equally likely pairs. Every other match eliminates one of the other players. Hence

$$p_2 = \prod_{k=3}^{9} \left(1 - \frac{2}{k(k-1)}\right) = \prod_{k=3}^{9} \frac{(k-2)(k+1)}{k(k-1)}.$$

The two factors telescope separately:

$$p_2 = \left(\frac{1}{2}\frac{2}{3}\cdots\frac{7}{8}\right)\left(\frac{4}{3}\frac{5}{4}\cdots\frac{10}{9}\right) = \frac{1}{8}\cdot\frac{10}{3} = \frac{5}{12}.$$

Now compare rank 2 with rank 3. Consider the first match involving two of the three strongest players. Until this match, all three are still present and each defeats any other player. At the stage when this first match occurs, each of the three pairs

$$(1,2), \qquad (1,3), \qquad (2,3)$$

is equally likely.

The first pair leaves ranks 1 and 3 present and eliminates rank 2. Each of the other two pairs leaves ranks 1 and 2 present and eliminates rank 3. In all three cases the surviving two are stronger than every other remaining player. From that point on, the probability that the weaker of those two reaches the final depends only on the number of remaining players, not on whether its original rank was 2 or 3.

This comparison holds after every possible history before the first match among the top three. Thus rank 2 reaches the final with twice the probability that rank 3 does. Writing the latter probability as p_3 gives

$$p_3 = \frac{p_2}{2} = \boxed{\frac{5}{24}}.$$

17. A

Initially write $P = (x, y)$. The reflections in the two axes are

$$A = (-x, y), \qquad B = (x, -y).$$

On the given parabola they coincide only at $P = (0, 0)$, which is excluded. Otherwise the line AB has equation

$$yX + xY = 0.$$

Let C be the reflection in $3X+2Y = 12$, and put $d = 3x+2y-12$. The segment PC is perpendicular to this line, so its direction is $(3, 2)$. Its midpoint lies on the line. Solving these two conditions gives

$$C = \left(x - \frac{6d}{13},\ y - \frac{4d}{13}\right).$$

Indeed, writing $C = (x + 3\lambda, y + 2\lambda)$ and imposing the midpoint condition gives $d + 13\lambda/2 = 0$.

For C to lie on AB, we require

$$y\left(x - \frac{6d}{13}\right) + x\left(y - \frac{4d}{13}\right) = 0.$$

This is equivalent to

$$13xy = (3y + 2x)(3x + 2y - 12).$$

Expanding and cancelling $13xy$ gives

$$x^2 + y^2 - 4x - 6y = 0.$$

Substituting $y = x^2$ now yields

$$x^4 - 5x^2 - 4x = x(x + 1)(x^2 - x - 4) = 0.$$

The root $x = 0$ gives the excluded origin. The three remaining roots are

$$-1, \qquad r = \frac{1 + \sqrt{17}}{2}, \qquad s = \frac{1 - \sqrt{17}}{2}.$$

They are distinct, real and nonzero.

The third reflection cannot coincide with either of the first two for these points. From the displayed coordinates, $C = A$ would require $P = (0, 6)$, and $C = B$ would require $P = (4, 0)$. Neither point lies on $y = x^2$. Thus all three retained roots give admissible points.

Since $r + s = 1$ and $rs = -4$, the requested sum is

$$-1 + \frac{1}{r} + \frac{1}{s} = -1 + \frac{r + s}{rs} = -1 - \frac{1}{4} = \boxed{-\frac{5}{4}}.$$

18. C

Suppose that such a pair exists, and label it so that $a < b$. Put $t = a + b$. Since $ab = 1$, we have

$$0 < a < 1 < b, \qquad t > 2.$$

Conversely, every $t > 2$ gives a distinct positive reciprocal pair: take the two roots of $z^2 - tz + 1 = 0$.

Factoring the difference of the two cubic values gives

$$0 = \frac{f_k(b) - f_k(a)}{b - a} = a^2 + ab + b^2 - k(a + b) + 10 = t^2 - kt + 9.$$

Thus the permitted pairs correspond exactly to the values $t > 2$ satisfying

$$k = t + \frac{9}{t}. \tag{1}$$

In particular,

$$k - 6 = \frac{(t - 3)^2}{t} \geq 0.$$

If $k < 6$, there are no pairs satisfying the hypothesis, so the stated property holds.

We next determine the signs of the tangent gradients. Put $r = 9/t$. By (1), $t + r = k$ and $tr + 1 = 10$, so

$$f_k(x) - f_k(a) = (x - a)(x - b)(x - r).$$

For example, expanding the right-hand side gives $x^3 - kx^2 + 10x - r$, and substituting $x = a$ shows $f_k(a) = r$. Differentiating the expanded polynomial and evaluating at the two roots gives

$$f_k'(a) = (a - b)(a - r), \qquad f_k'(b) = (b - a)(b - r). \tag{2}$$

Suppose $6 \leq k < 13/2$. The quadratic $t^2 - kt + 9$ is positive at $t = 2$ and non-positive at $t = 3$, since these values are $13 - 2k > 0$ and $18 - 3k \leq 0$. Hence it has a root with $2 < t \leq 3$. Choose the reciprocal pair belonging to this root. Then

$$b < t \leq 3 \leq r,$$

so $f_k'(b) < 0$ by (2). This supplies a counterexample to the property for every k in this range, including $k = 6$.

Finally, suppose $k \geq 13/2$, and consider any permitted pair. From (1),

$$k - \frac{13}{2} = \frac{(t - 2)(2t - 9)}{2t}.$$

As $t > 2$, this forces $t \geq 9/2$. Consequently

$$r = \frac{9}{t} \leq 2, \qquad b = t - a > t - 1 \geq \frac{7}{2},$$

so $r < b$. Also,

$$r - a = \frac{9 - a(a + b)}{t} = \frac{8 - a^2}{t} > 0,$$

because $0 < a < 1$. Thus $a < r < b$, and both gradients in (2) are strictly positive.

At $k = 13/2$, the two possible values of t from (1) are 2 and $9/2$. The value $t = 2$ would give $a = b = 1$ and is excluded; the other value works. Therefore the necessary and sufficient condition is

$$\boxed{k < 6 \quad \text{or} \quad k \geq \frac{13}{2}}.$$

S therefore implies T. To test necessity, take $k = 0$. There is no admissible pair, because every such pair would require $k = t + 9/t \geq 6$. Thus T, which is a universal conditional statement, is true while S is false. No existence condition was imposed in T.

Hence S is sufficient but not necessary for T, giving C.

19. D

Removing square factors from the labels leaves four types, each occurring twice:

type	1	2	3	6
labels	$1, 4$	$2, 8$	$3, 12$	$6, 24.$

A product is a square precisely when the powers of both 2 and 3 are even. Therefore a face is permitted precisely when it contains either two complete pairs of types or one of each of the four types. Indeed, the product of all four types is 36, while the product of any two different types is not a square. A face with one repeated type and two different remaining types cannot work.

First count all permitted placements. Write the vertices of one face in cyclic order as A, B, C, D, and the vertices directly above them as E, F, G, H. Fix the label 1 at A. There are three kinds of possible position for its partner 4.

- If 4 is at B, the faces $ABCD$ and $ABFE$ force C, D to be a pair of one type, and E, F to be a pair of another type. Then G, H are the third pair. The other edge positions work in exactly the same way.
- If 4 is at C, the face $ABCD$ forces B, D to be a pair. The side faces then force E, G to be one pair and F, H to be the other. The other face-diagonal positions work in the same way.
- If 4 is at G, each face containing A must contain all four types. Thus B, D, E have three different types. The faces through A then force C to have the type of E, F the type of D, and H the type of B.

In every case there are 3! ways to assign the three remaining types to their three pairs, and 2^3 ways to place their two distinct labels within those pairs. Each resulting placement satisfies all six face conditions: every face has two complete pairs or all four types. Thus every one of the seven possible positions of 4 allows exactly $3!\,2^3$ placements. Restoring the eight choices for the vertex carrying 1 gives

$$8 \cdot 7 \cdot 3! \cdot 2^3 = 2688$$

permitted placements.

Now fix a face to contain $1, 2, 3, 6$. Its labels can be placed in 4! orders. For any such order, write their types around the face as u, v, w, z. The opposite face contains the remaining label of each type. Its cyclic type order, with corresponding vertices directly above one another, can be exactly one of

$$(u, v, w, z), \quad (v, u, z, w), \quad (w, z, u, v), \quad (z, w, v, u).$$

To see completeness, choose the type above u. On a side face, its two lower types are different; the square-product condition then uniquely determines the other upper type. Continuing around the cube forces the listed order. Each of the four orders works.

There are consequently $4! \cdot 4 = 96$ favourable placements for each chosen face. The same four labels cannot occupy two different faces, so the six face choices give $6 \cdot 96 = 576$ distinct favourable placements.

The required conditional probability is

$$\frac{576}{2688} = \boxed{\frac{3}{14}}.$$

20. E

Write the four numbers in increasing order as

$$t-3d, \quad t-d, \quad t+d, \quad t+3d, \qquad d>0.$$

They must split into two pairs each having sum of squares 1, because

$$\sin^2\alpha+\cos^2\alpha=\sin^2\beta+\cos^2\beta=1.$$

There are three ways to split four ordered entries into two pairs. Pairing the two outer entries together is impossible: their sum of squares exceeds that of the two inner entries by

$$(t-3d)^2+(t+3d)^2-(t-d)^2-(t+d)^2=16d^2>0.$$

For the other two pairings, the differences between the sums of squares are respectively

$$(t-3d)^2+(t-d)^2-(t+d)^2-(t+3d)^2=-16td,$$
$$(t-3d)^2+(t+d)^2-(t-d)^2-(t+3d)^2=-8td.$$

Equality therefore forces $t=0$. Each admissible pair then contains one entry of magnitude d and one of magnitude $3d$. Its sum of squares is $10d^2=1$, so the four entries are exactly

$$-\frac{3}{\sqrt{10}}, \quad -\frac{1}{\sqrt{10}}, \quad \frac{1}{\sqrt{10}}, \quad \frac{3}{\sqrt{10}}.$$

Write $(\sin\alpha,\cos\alpha)=(u,v)$. The ratio u/v has magnitude 3 or $1/3$, and the two unused entries are $-u$ and $-v$. Consequently

$$(\sin\beta,\cos\beta)=(-u,-v) \quad \text{or} \quad (-v,-u).$$

In the first case the tangent product is $(u/v)^2$, giving 9 or $1/9$; in the second case it is 1. There are no further possibilities.

All three values are attained. For example, the respective ordered pairs of coordinate numerators

$$(3,1;\,-3,-1), \qquad (1,3;\,-1,-3), \qquad (3,1;\,-1,-3),$$

after division by $\sqrt{10}$, give unit-circle coordinates for $(\sin\alpha,\cos\alpha;\sin\beta,\cos\beta)$ and produce 9, $1/9$ and 1.

The requested sum of distinct values is therefore

$$9+\frac{1}{9}+1=\boxed{\frac{91}{9}}.$$