1. The quadratic polynomials f and g both have coefficient 1 for x^2. For every real number x,

$$f(g(x)) - g(f(x)) = 4x^2 - 4x + 3.$$

What is the least possible value of $f(x) + g(x)$ as x varies over the real numbers?

A 1
B 2
C $\frac{5}{2}$
D 3
E $\frac{7}{2}$
F 4

2. Two positive integers a and b, with $a < b$, satisfy

$$a + b = 120.$$

The sum of their highest common factor and their lowest common multiple is 360.

What is the sum of all the distinct possible values of $b - a$?

A 76

B 20

C 40

D 56

E 60

F 152

3. Four non-negative real numbers have mean 1. The mean of their squares is $\frac{3}{2}$.

What is the greatest possible median of the four numbers?

A 1

B $\frac{7}{6}$

C $\frac{\sqrt{6}}{2}$

D $\frac{5}{4}$

E $\frac{4}{3}$

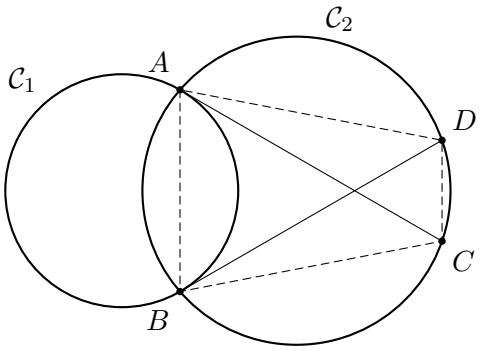
F $\frac{3}{2}$

4. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have equations

$$\mathcal{C}_1 : \ x^2 + y^2 = 4, \qquad \mathcal{C}_2 : \ (x-3)^2 + y^2 = 7.$$

They intersect at A and B, with A above B. The tangent to $\mathcal{C}_1$ at A meets $\mathcal{C}_2$ again at C, and the tangent to $\mathcal{C}_1$ at B meets $\mathcal{C}_2$ again at D.

What is the area of quadrilateral $ABCD$?



A $\dfrac{9\sqrt{3}}{4}$

B $3\sqrt{3}$

C $\dfrac{27\sqrt{3}}{4}$

D $\dfrac{9\sqrt{3}}{2}$

E $9\sqrt{3}$

F $\dfrac{27\sqrt{3}}{2}$

5. A drone's path relative to the ground follows the four equal sides of a square, travelling east, north, west and south in that order. The wind velocity is constant throughout the flight, and the drone keeps the same speed relative to the air on every side.

Its speeds relative to the ground on the east, north and south sides are 10, 8 and 2 metres per second, respectively.

The drone's velocity relative to the ground is the vector sum of its velocity relative to the air and the wind velocity.

Ignoring time spent turning at the corners, what is the drone's average speed relative to the ground for the whole circuit, in metres per second?

A $\frac{8}{5}$

B $\frac{80}{29}$

C $\frac{16}{5}$

D $\frac{120}{29}$

E $\frac{27}{5}$

F $\frac{80}{27}$

6. Two distinct points P and Q lie on the curve

$$y = x^3 - 2x.$$

The tangents to the curve at P and Q are perpendicular, and their point of intersection lies on the x-axis.

What is the product of the x-coordinates of P and Q?

A $-\frac{3}{2}$

B $-\frac{7}{6}$

C -1

D $-\frac{5}{6}$

E $-\frac{2}{3}$

F $-\frac{1}{3}$

7. The positive real numbers a, b, c satisfy $a \leq b \leq c$. Both

$$a,\ b,\ c \qquad \text{and} \qquad \frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}$$

can be used as the side lengths of triangles of positive area.

Writing $R = c/a$, which of the following is the complete set of possible values of R?

A $1 \leq R < 2$

B $1 \leq R \leq \dfrac{3+\sqrt{5}}{2}$

C $1 \leq R < \dfrac{3+\sqrt{5}}{2}$

D $1 < R < \dfrac{3+\sqrt{5}}{2}$

E $1 \leq R < 1+\sqrt{2}$

F $1 \leq R < \dfrac{1+\sqrt{5}}{2}$

8. A real constant c is to be chosen so that three distinct points common to the curves

$$y = 2^x \qquad \text{and} \qquad y = (x - c)^2$$

have x-coordinates which, in increasing order, form an arithmetic progression.

Which of the following is possible for the common difference of this progression?

A $\log_2(1 + \sqrt{2})$

B $\log_2 3$

C $\log_2(2 + \sqrt{2})$

D $\log_2(3 + 2\sqrt{2})$

E $\log_2(17 + 12\sqrt{2})$

F No such choice of c is possible

9. For a real constant k, the equation

$$\sin^3\theta + \cos^3\theta = k$$

has exactly three distinct solutions $\theta_1, \theta_2, \theta_3$ with $0 \leq \theta < 2\pi$.

What is the value of

$$\sum_{i=1}^{3} (\sin\theta_i + \cos\theta_i)^2 \, ?$$

A $3 - \sqrt{3}$

B $4 - \sqrt{3}$

C 3

D $2\sqrt{3}$

E $6 - 2\sqrt{3}$

F 4

10. Initially a list has the single entry $\frac{1}{2}$. In each round, every entry x is replaced simultaneously by the two entries

$$\frac{x}{3} \quad \text{and} \quad \frac{x+2}{3}.$$

After eight rounds the list has 256 entries. For each entry x, the value $x(1-x)$ is calculated.

What is the sum of these 256 values?

A $16(1+3^{-16})$

B $32(1+3^{-16})$

C $32(1-3^{-16})$

D $32(1+3^{-8})$

E $32(1-3^{-8})$

F $16(1+3^{-8})$

11. Ten cards, six red and four blue, are shuffled uniformly at random into a row. It is given that no two blue cards are next to one another.

What is the probability that every red card is next to at least one blue card?

A $\frac{1}{21}$

B $\frac{3}{35}$

C $\frac{2}{7}$

D $\frac{2}{5}$

E $\frac{18}{35}$

F $\frac{5}{7}$

12. The lines

$$L_1 : \ y = mx + 1, \qquad L_2 : \ y = mx - 2$$

each meet the curve $y = x^2$ at two distinct points. The points on L_1 are A and B, and those on L_2 are C and D.

The area bounded by the segment AB and the curve is eight times the area bounded by the segment CD and the curve.

What is the value of $AB \cdot CD$?

A 8

B $8\sqrt{13}$

C 52

D 208

E 104

F 416

13. The curve

$$y = x^4 - 3x^2 + 1$$

meets the x-axis at four distinct points.

The gradients of the normals at these points are n_1, n_2, n_3, n_4.

What is the value of

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 \,?$$

A $\frac{3}{10}$

B $\frac{3}{20}$

C $\frac{1}{10}$

D $\frac{3}{5}$

E $\frac{3}{2}$

F $\frac{6}{5}$

14. For $r = 0, 1, \ldots, 30$, let a_r be the coefficient of x^r in the expansion of $(1 + x)^{30}$.

What is the value of

$$a_0 + a_3 + a_6 + \cdots + a_{30}\,?$$

A $\dfrac{2^{30} - 2^{10}}{3}$

B $\dfrac{2^{30} - 4}{3}$

C $\dfrac{2^{30} - 1}{3}$

D $\dfrac{2^{30} + 2}{3}$

E $\dfrac{2^{30} + 8}{3}$

F $\dfrac{2^{30} + 2^{11}}{3}$

15. The points A and B lie on the curve

$$y = x^2,$$

and the length of the straight line segment AB is 4.

What is the least possible y-coordinate of the midpoint of AB?

A $\frac{3}{4}$

B 1

C $\frac{7}{4}$

D 2

E 3

F 4

16. The terms of a geometric progression are positive. Its first three terms, when rounded to the nearest integer, are 2, 3 and 5, in that order. An exact half-integer is rounded upwards.

What is the sum of all the integers to which the fourth term can round?

A 40

B 45

C 51

D 56

E 57

F 63

17. A particle moves along a straight line. At time t seconds, its signed displacement in metres from its starting point is

$$s(t) = t^3 - 6t^2 + bt, \qquad 0 \leq t \leq 4,$$

where b is a real constant.

It is given that $s(4) > 0$ and that the total distance travelled during these four seconds is $3s(4)$.

At what time does the particle first return to its starting point after $t = 0$?

A 1 second

B $\frac{3}{2}$ seconds

C 2 seconds

D $\frac{8}{3}$ seconds

E 3 seconds

F There is no return for $0 < t \leq 4$

18. Ten counters numbered $1, 2, \ldots, 10$ are divided at random into five pairs, with every division into pairs equally likely.

It is known that the five pair sums are distinct odd numbers which, when written in increasing order, form an arithmetic progression.

What is the probability that counters 1 and 2 are in the same pair?

A $\frac{1}{9}$

B $\frac{1}{7}$

C $\frac{1}{6}$

D $\frac{1}{5}$

E $\frac{1}{4}$

F $\frac{1}{2}$

19. The nine entries in the array

$$\begin{array}{ccc} 1 & a & b \\ c & d & e \\ f & g & 225 \end{array}$$

are distinct positive integers.

In each row, the entries from left to right form an arithmetic progression. In each column, the entries from top to bottom form a geometric progression.

What is the value of d?

A 13

B 15

C 25

D 39

E 75

F 117

20. A square has its centre at the origin and all four of its vertices on the curve

$$y = x^3 - 3x.$$

What is the sum of all the distinct values that the area of the square can take?

A 8

B 9

C 10

D 16

E 18

F 36