1. D

Write

$$f(x) = x^2 + ax + b, \qquad g(x) = x^2 + cx + d.$$

The coefficient of x^3 in $f(g(x)) - g(f(x))$ is $2(c - a)$. The displayed identity therefore forces $c = a$. With this substitution, expansion and factorisation give

$$f(g(x)) - g(f(x)) = (d - b)\big(2x^2 + 2ax + b + d + a - 1\big).$$

Comparing the coefficients of x^2, x and the constant terms with $4x^2 - 4x + 3$ gives, in that order,

$$d - b = 2, \qquad a = -1, \qquad b + d = \frac{7}{2}.$$

Consequently

$$f(x) + g(x) = 2x^2 - 2x + \frac{7}{2} = 2\left(x - \frac{1}{2}\right)^2 + 3.$$

Its least value is $\boxed{3}$, attained at $x = 1/2$. The polynomial conditions are consistent: $b = 3/4$ and $d = 11/4$ give the required identity.

2. A

Let h be the highest common factor, and write

$$a = hu, \qquad b = hv,$$

where $u < v$ are coprime positive integers. Their lowest common multiple is huv, so the two given conditions imply

$$h(u+v) = 120, \qquad h(uv+1) = 360.$$

Dividing gives $uv + 1 = 3(u+v)$, or

$$(u-3)(v-3) = 8.$$

If both factors were negative, then u, v would be positive integers smaller than 3, so their product on the left would be at most 4. The factors must therefore both be positive. The positive factor pairs of 8, in increasing order, are $(1, 8)$ and $(2, 4)$. They give

$$(u, v) = (4, 11) \quad \text{or} \quad (5, 7).$$

Both pairs are coprime. The first gives $h = 120/15 = 8$ and $(a, b) = (32, 88)$; the second gives $h = 120/12 = 10$ and $(a, b) = (50, 70)$. These do satisfy the original conditions. The two distinct differences are 56 and 20, so their sum is $\boxed{76}$.

3. B

Write the numbers in order as $a \le b \le c \le d$, and let their median be m. Then

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2=6, \qquad b+c=2m.$$

We show that m cannot exceed $7/6$. Suppose that $m > 7/6$, so in particular $m > 1$. Since $c \ge m$, $d \ge c$ and $a \ge 0$, the sum of the four numbers is at least $3m$. Hence $m \le 4/3$.

For a fixed sum and median, the sum of squares can only increase if we replace

$$(a,b,c,d) \quad \text{by} \quad (0, b-a, c+a, d+a).$$

The new numbers are still non-negative and ordered, and their sum and median are unchanged. The increase in the sum of squares is

$$2a(c+d-b)+2a^2 \ge 0.$$

After this replacement the largest number is $4-2m$. The two middle numbers have sum $2m$. Moving an amount from the smaller of these to the larger increases the sum of their squares. Their largest possible upper value is $4-2m$, because it cannot exceed the largest number. This is attainable: since $1 < m \le 4/3$, the resulting ordered list is

$$0, \quad 4m-4, \quad 4-2m, \quad 4-2m.$$

Therefore any original list with such a median satisfies

$$6 \le (4m-4)^2+2(4-2m)^2 = 6+24\left(m-\frac{7}{6}\right)\left(m-\frac{3}{2}\right).$$

But the final product is negative when $7/6 < m \le 4/3$, a contradiction. Thus $m \le 7/6$.

The list

$$0, \quad \frac{2}{3}, \quad \frac{5}{3}, \quad \frac{5}{3}$$

has sum 4, sum of squares 6 and median $7/6$. The upper bound is attained, so the greatest possible median is $\boxed{7/6}$.

4. C

Subtracting the circle equations at a common point gives

$$-6x + 9 = 3,$$

so $x = 1$. It follows that

$$A = (1, \sqrt{3}), \qquad B = (1, -\sqrt{3}).$$

The radius from the origin to A has gradient $\sqrt{3}$, so the tangent at A has gradient $-1/\sqrt{3}$. Its equation is

$$x + \sqrt{3}\, y = 4.$$

Substitute $y = (4 - x)/\sqrt{3}$ into the equation of $\mathcal{C}_2$:

$$(x - 3)^2 + \frac{(4 - x)^2}{3} = 7, \qquad 4x^2 - 26x + 22 = 2(x - 1)(2x - 11) = 0.$$

The root $x = 1$ is A, and the other root gives

$$C = \left(\frac{11}{2}, -\frac{\sqrt{3}}{2} \right).$$

Reflection in the x-axis gives

$$D = \left(\frac{11}{2}, \frac{\sqrt{3}}{2} \right).$$

Thus $ABCD$ is a trapezium. Its parallel sides have lengths $2\sqrt{3}$ and $\sqrt{3}$, and the perpendicular distance between them is $11/2 - 1 = 9/2$. Its area is

$$\frac{1}{2}(2\sqrt{3} + \sqrt{3})\frac{9}{2} = \boxed{\frac{27\sqrt{3}}{4}}.$$

5. F

Take east and north as the two positive coordinate directions. Write the wind velocity as (u, v) and the drone's constant speed relative to the air as s.

If its ground velocity is (x, y), then its velocity relative to the air is $(x - u, y - v)$. Pythagoras' theorem gives

$$(x-u)^2 + (y-v)^2 = s^2,$$

or

$$x^2 + y^2 - 2ux - 2vy + K = 0, \qquad K = u^2 + v^2 - s^2.$$

For the north and south sides, $x = 0$ and the two values of y are 8 and -2. Hence

$$y^2 - 2vy + K = (y-8)(y+2), \qquad K = -16.$$

Let the westward ground speed be $w > 0$. For the east and west sides, $y = 0$ and the two values of x are 10 and $-w$. Therefore

$$x^2 - 2ux + K = (x-10)(x+w), \qquad K = -10w.$$

The same constant K occurs in both quadratics. Thus $10w = 16$ and $w = 8/5$.

If the side length of the square is L metres, the total time in seconds is

$$\frac{L}{10} + \frac{L}{8} + \frac{5L}{8} + \frac{L}{2} = \frac{27L}{20}.$$

The total distance is $4L$, so the required average speed is

$$\frac{4L}{27L/20} = \boxed{\frac{80}{27}} \text{ metres per second.}$$

The given speeds are compatible with a steady wind and a fixed airspeed: coefficient comparison gives $u = 21/5$, $v = 3$ and $s^2 = 1066/25$.

6. D

Let the two x-coordinates be p and q, with $p \neq q$. The tangent at input t is

$$y = (3t^2 - 2)x - 2t^3.$$

Both tangent gradients are finite. Since the lines are perpendicular, neither gradient can be zero and

$$(3p^2 - 2)(3q^2 - 2) = -1. \tag{1}$$

Their common intersection is on the x-axis, so their x-intercepts agree:

$$\frac{2p^3}{3p^2 - 2} = \frac{2q^3}{3q^2 - 2}.$$

Cross-multiplying and factoring gives

$$(p - q)(3p^2q^2 - 2(p^2 + pq + q^2)) = 0.$$

As $p \neq q$, we obtain

$$3p^2q^2 = 2(p^2 + pq + q^2). \tag{2}$$

Expanding (1), and then using three times (2), gives

$$-1 = 9p^2q^2 - 6(p^2 + q^2) + 4 = 6pq + 4.$$

Therefore $pq = \boxed{-5/6}$. The conditions are attainable. For example, choose p, q to have sum $\sqrt{5/24}$ and product $-5/6$. They are distinct real roots of a quadratic, and $p^2 + q^2 = 15/8$. These values satisfy (1) and (2); (1) ensures the nonzero denominators, and (2) ensures equal x-intercepts.

7. C

Scale the first three lengths by $1/a$, and write $s = b/a$. Then $1 \le s \le R$. A positive ordered triple forms a triangle of positive area exactly when its largest length is smaller than the sum of the other two. Thus the two triangle conditions become

$$R < 1 + s, \qquad 1 < \frac{1}{s} + \frac{1}{R}.$$

The second inequality follows after scaling the reciprocal lengths by a. If $R > 1$, these inequalities imply

$$R - 1 < s < \frac{R}{R-1}.$$

Consequently

$$(R-1)^2 < R, \qquad R^2 - 3R + 1 < 0.$$

Since $R > 1$, this yields $R < (3 + \sqrt{5})/2$. The value $R = 1$ is also possible, by taking all three lengths equal.

To establish that every value below the upper bound occurs, take

$$a = 1, \qquad b = \sqrt{R}, \qquad c = R$$

for any $1 \le R < (3 + \sqrt{5})/2$. Multiplying the reciprocal lengths by R gives exactly the same three lengths in reverse order. It therefore suffices to check

$$R < 1 + \sqrt{R}.$$

On putting $t = \sqrt{R} \ge 1$, this becomes $t^2 - t - 1 < 0$, which holds precisely when

$$t < \frac{1 + \sqrt{5}}{2}, \qquad \text{or equivalently} \qquad R < \frac{3 + \sqrt{5}}{2}.$$

Both triangles have positive area for these choices. At the excluded upper endpoint the triangle inequalities cannot both be strict. The complete set is therefore

$$\boxed{1 \le R < \frac{3 + \sqrt{5}}{2}}.$$

8. D

Write the three x-coordinates as $a, a+d, a+2d$, where $d > 0$, and put

$$s = 2^{a/2} > 0, \qquad r = 2^{d/2} > 1.$$

At the three points, the absolute values of $x - c$ are respectively

$$s, \qquad sr, \qquad sr^2.$$

The signed values $a - c, a + d - c, a + 2d - c$ are strictly increasing. The first two cannot both be negative: their absolute values would then decrease, whereas $sr > s$. Thus the middle and last signed values are positive.

If the first were also positive, equal horizontal gaps would give

$$sr - s = sr^2 - sr,$$

so $s(r-1) = sr(r-1)$, which is impossible for $s > 0$ and $r > 1$. No signed value is zero, since $2^x > 0$. The signed values must therefore be

$$-s, \qquad sr, \qquad sr^2.$$

Equating the two gaps now gives

$$s(r+1) = sr(r-1), \qquad r^2 - 2r - 1 = 0.$$

Since $r > 1$, we have $r = 1 + \sqrt{2}$. Hence

$$d = 2\log_2 r = \log_2(r^2) = \boxed{\log_2(3 + 2\sqrt{2})}.$$

This value is attainable. Take $r = 1 + \sqrt{2}$, $d = 2\log_2 r$, and define

$$s = \frac{d}{1+r}, \qquad a = 2\log_2 s, \qquad c = a + s.$$

Then $s > 0$ and $r^2 = 2r + 1$, so the signed distances from c at $a, a+d, a+2d$ are exactly $-s, sr, sr^2$. Their squares are s^2, s^2r^2, s^2r^4, which are also $2^a, 2^{a+d}, 2^{a+2d}$. Thus all three points lie on both curves, and they are distinct because $d > 0$.

9. E

Write $s = \sin\theta + \cos\theta$. Then

$$\sin\theta\cos\theta = \frac{s^2-1}{2}, \qquad \sin^3\theta + \cos^3\theta = s^3 - 3s\frac{s^2-1}{2} = \frac{3s-s^3}{2}.$$

The points $(\cos\theta, \sin\theta)$ go once around the unit circle as $0 \le \theta < 2\pi$. The line whose coordinates sum to s meets this circle twice when $-\sqrt{2} < s < \sqrt{2}$, once when $s = \pm\sqrt{2}$, and not at all otherwise. These claims also follow by solving $u + v = s$, $u^2 + v^2 = 1$.

Consequently, every admissible interior value of s contributes two angles. An odd total of three angles is possible only if $s = \sqrt{2}$ or $s = -\sqrt{2}$ is a solution. Substitution gives

$$k = \frac{\sqrt{2}}{2} \quad \text{or} \quad k = -\frac{\sqrt{2}}{2}.$$

Replacing an angle by the angle half a turn further round negates s and k, but leaves s^2 unchanged. It is therefore enough to consider $k = \sqrt{2}/2$.

For this value,

$$s^3 - 3s + \sqrt{2} = (s - \sqrt{2})(s^2 + \sqrt{2}s - 1) = 0.$$

Besides $s = \sqrt{2}$, the roots are

$$\frac{\sqrt{6}-\sqrt{2}}{2} \quad \text{and} \quad -\frac{\sqrt{6}+\sqrt{2}}{2}.$$

The first lies strictly between 0 and $\sqrt{2}$, and the second is less than $-\sqrt{2}$. Thus there is one angle with $s = \sqrt{2}$ and two with $s = (\sqrt{6} - \sqrt{2})/2$, confirming exactly three solutions. Their squared values sum to

$$2 + 2\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 = 2 + 2(2 - \sqrt{3}) = \boxed{6 - 2\sqrt{3}}.$$

10. B

Let $g(x) = x(1-x)$, and let S_n be the sum of $g(x)$ over the list after n rounds. In particular, $S_0 = 1/4$.

The two entries replacing x contribute

$$g\left(\frac{x}{3}\right) + g\left(\frac{x+2}{3}\right) = \frac{3x - x^2 + (x+2)(1-x)}{9}$$
$$= \frac{2}{9}(1 + x - x^2) = \frac{2}{9}(1 + g(x)).$$

There are 2^n entries after n rounds, so

$$S_{n+1} = \frac{2}{9}(2^n + S_n).$$

Set $A_n = S_n/2^n$, the mean of the values at that stage. Dividing by 2^{n+1} gives

$$A_{n+1} = \frac{1 + A_n}{9}, \qquad A_{n+1} - \frac{1}{8} = \frac{1}{9}\left(A_n - \frac{1}{8}\right).$$

Since $A_0 - 1/8 = 1/8$, this is a geometric sequence:

$$A_n = \frac{1}{8} + \frac{1}{8 \cdot 9^n}.$$

Hence

$$S_8 = 2^8 A_8 = \boxed{32(1 + 3^{-16})}.$$

11. C

Every colour sequence with six red and four blue cards is equally likely: it comes from the same number 6!4! of arrangements of individually labelled cards.

First place the six red cards in a row. There are seven gaps, including the two end gaps, in which a blue card can be inserted. No two blue cards are adjacent precisely when the four blue cards occupy four different gaps. There are therefore

$$\binom{7}{4} = 35$$

colour sequences under the given condition.

For the favourable sequences, instead use the four blue cards to mark five gaps for red cards. Each of the three internal gaps must contain at least one red card. To ensure that every red card has a blue neighbour, each internal gap can contain at most two red cards, while each end gap can contain at most one.

Place one red card in each internal gap. Three red cards remain, and each of the five gaps can now receive at most one more. Choosing the three gaps that receive these remaining cards gives

$$\binom{5}{3} = 10$$

favourable sequences. These are distinct and satisfy both conditions, so the required conditional probability is

$$\boxed{\frac{10}{35} = \frac{2}{7}}.$$

12. E

For the line $y = mx + c$, the vertical separation above the parabola is

$$mx + c - x^2 = \left(\frac{m^2}{4} + c\right) - \left(x - \frac{m}{2}\right)^2.$$

Write $h = m^2/4 + c$. Two distinct intersection points mean $h > 0$. Translating the horizontal coordinate by $m/2$, the area of the bounded region is

$$\int_{-\sqrt{h}}^{\sqrt{h}} (h - u^2)\,\mathrm{d}u = \frac{4}{3}h^{3/2}.$$

For L_1 and L_2, put

$$h_1 = \frac{m^2}{4} + 1, \qquad h_2 = \frac{m^2}{4} - 2.$$

Both are positive, $h_1 - h_2 = 3$, and the area ratio gives

$$\left(\frac{h_1}{h_2}\right)^{3/2} = 8, \qquad h_1 = 4h_2.$$

Hence $h_2 = 1$, $h_1 = 4$ and $m^2 = 12$.

The horizontal distances between the endpoints of the two chords are $2\sqrt{h_1} = 4$ and $2\sqrt{h_2} = 2$. A segment on a line of gradient m with horizontal length d has length $d\sqrt{1 + m^2}$, by Pythagoras. Therefore

$$AB \cdot CD = 4\sqrt{13} \cdot 2\sqrt{13} = \boxed{104}.$$

The values $m = \pm 2\sqrt{3}$ both give the stated two intersections and area ratio, so the conditions are attainable.

13. A

At an x-intercept, put $u = x^2$. Then

$$u^2 - 3u + 1 = 0.$$

Its two roots u_1, u_2 are positive, since they are $(3 \pm \sqrt{5})/2$, and satisfy $u_1 + u_2 = 3$ and $u_1 u_2 = 1$. Each gives two intercepts, at $x = \pm\sqrt{u_i}$.

The tangent gradient is

$$4x^3 - 6x = 2x(2u - 3).$$

Using $u^2 = 3u - 1$,

$$(2u - 3)^2 = 4u^2 - 12u + 9 = 5.$$

Thus the square of the tangent gradient is $20u$, which is nonzero. The square of the normal gradient is consequently $1/(20u)$.

Each of u_1, u_2 contributes twice, so the required sum is

$$\frac{1}{10}\left(\frac{1}{u_1} + \frac{1}{u_2}\right) = \frac{1}{10}\frac{u_1 + u_2}{u_1 u_2} = \boxed{\frac{3}{10}}.$$

14. D

For the expansion of $(1+x)^{3m}$, let A_m, B_m, C_m be the sums of the coefficients whose exponents leave remainders $0, 1, 2$, respectively, when divided by 3.

Multiplying by

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

shows that

$$\begin{aligned} A_{m+1} &= 2A_m + 3B_m + 3C_m, \\ B_{m+1} &= 3A_m + 2B_m + 3C_m, \\ C_{m+1} &= 3A_m + 3B_m + 2C_m. \end{aligned}$$

For example, the terms 1 and x^3 preserve an exponent's remainder, while multiplication by x or x^2 changes it by 1 or 2.

Subtracting the equations gives

$$A_{m+1} - B_{m+1} = -(A_m - B_m), \qquad B_{m+1} - C_{m+1} = -(B_m - C_m).$$

Initially $A_0 = 1$ and $B_0 = C_0 = 0$. Hence $B_m = C_m$, while $A_m - B_m$ alternates between 1 and -1. In particular, at $m = 10$,

$$A_{10} = B_{10} + 1 = C_{10} + 1.$$

The sum of all coefficients is $(1+1)^{30} = 2^{30}$. Therefore

$$2^{30} = A_{10} + B_{10} + C_{10} = 3A_{10} - 2,$$

and the required sum is

$$\boxed{\frac{2^{30}+2}{3}}.$$

15. C

Write the two points as

$$A = (u-h, (u-h)^2), \qquad B = (u+h, (u+h)^2), \qquad h > 0.$$

Their midpoint is (u, u^2+h^2). The differences between the endpoint coordinates are $2h$ and $4uh$, so Pythagoras gives

$$16 = AB^2 = 4h^2 + 16u^2h^2 = 4h^2(1+4u^2).$$

Thus $h^2 = 4/(1+4u^2)$.

Put $v = 1+4u^2$, so $v \geq 1$. The midpoint's height is

$$u^2+h^2 = \frac{v-1}{4} + \frac{4}{v} = \frac{7}{4} + \frac{(v-4)^2}{4v} \geq \frac{7}{4}.$$

Equality is possible: take $u = \sqrt{3}/2$ and $h = 1$. These values give $AB = 4$ and midpoint height $3/4 + 1 = 7/4$.

The least height is therefore $\boxed{7/4}$. A horizontal chord would instead have endpoints $(-2, 4)$ and $(2, 4)$; restricting attention to that symmetric position would miss the minimum.

16. C

Let the first, third and fourth terms be a, c and w. Positivity and the geometric-progression rule give

$$\text{second term} = \sqrt{ac}, \qquad w^2 = \frac{c^3}{a}.$$

The first and third rounding conditions imply

$$\frac{3}{2} \le a < \frac{5}{2}, \qquad \frac{9}{2} \le c < \frac{11}{2}.$$

Consequently

$$\frac{729}{20} < w^2 < \frac{1331}{12}.$$

The exact comparisons

$$\left(\frac{11}{2}\right)^2 < \frac{729}{20} < \left(\frac{13}{2}\right)^2, \qquad \left(\frac{21}{2}\right)^2 < \frac{1331}{12} < \left(\frac{23}{2}\right)^2$$

show that only $6, 7, 8, 9, 10, 11$ can be obtained by rounding w.

All six possibilities really occur under all three rounding conditions. For $0 < t < 1$, choose

$$a = \frac{5}{2} - t, \qquad c = \frac{9}{2} + t,$$

and use the positive common ratio $\sqrt{c/a}$. The second term satisfies

$$\left(\frac{5}{2}\right)^2 < \frac{33}{4} < ac = \frac{45}{4} - 2t - t^2 < \frac{45}{4} < \left(\frac{7}{2}\right)^2.$$

Thus the first three terms round to $2, 3, 5$ throughout this family. The fourth term increases continuously as t increases, approaching

$$\sqrt{\frac{729}{20}} \quad \text{as } t \text{ approaches } 0, \qquad \sqrt{\frac{1331}{12}} \quad \text{as } t \text{ approaches } 1.$$

The first limit is strictly between $11/2$ and $13/2$, and the second is strictly between $21/2$ and $23/2$. Hence every rounded integer from 6 to 11 occurs.

Their sum is $6 + 7 + 8 + 9 + 10 + 11 = \boxed{51}$.

17. E

Put $D = s(4) = 4b - 32 > 0$, so $b > 8$. The velocity is

$$s'(t) = 3t^2 - 12t + b = 3(t-2)^2 + b - 12.$$

If $b \geq 12$, the particle never reverses direction and its total distance is D, contrary to the given total $3D$. Thus $8 < b < 12$.

There are therefore two stationary times

$$t = 2 - \sqrt{4 - b/3}, \qquad t = 2 + \sqrt{4 - b/3},$$

both strictly between 0 and 4. The particle first moves in the positive direction, then in the negative direction, and finally in the positive direction. Let its displacement at the first turning point be M and at the second be m.

Its total distance is

$$M + (M - m) + (D - m) = D + 2(M - m).$$

The given condition gives $M - m = D$. On the other hand, expansion shows that

$$s(t) + s(4 - t) = D.$$

The two stationary times sum to 4, so $M + m = D$. The two equations imply $m = 0$.

Thus the particle returns to its starting point at the second turning point. If this time is $T > 0$, then $s(T) = s'(T) = 0$, giving

$$T^2 - 6T + b = 0, \qquad 3T^2 - 12T + b = 0.$$

Subtracting gives $2T(T - 3) = 0$, so $T = 3$.

For verification, this forces $b = 9$ and

$$s(t) = t(t-3)^2.$$

The only zeros for $0 \leq t \leq 4$ are 0 and 3. The successive displacement values are $0, 4, 0, 4$ at times $0, 1, 3, 4$, so the total distance is $12 = 3s(4)$. The first return after the start is at $\boxed{\text{3 seconds}}$.

18. B

The sum of the five pair sums is $1+2+\cdots+10=55$. Their middle term is therefore 11. Write the increasing progression as

$$11-2d, \quad 11-d, \quad 11, \quad 11+d, \quad 11+2d.$$

The sums are distinct and odd, so d is a positive even integer. Every pair sum is at least 3 and at most 19, giving $d \leq 4$. Hence $d=2$ or $d=4$.

If $d=4$, the sums are $3, 7, 11, 15, 19$. The extreme sums first force $(1,2)$ and $(9,10)$; the remaining extreme sums then force $(3,4)$ and $(7,8)$, leaving $(5,6)$. There is exactly one division of this kind.

If $d=2$, the sums are $7, 9, 11, 13, 15$. The pair summing to 7 must be $(1,6)$, $(2,5)$ or $(3,4)$. For each choice, testing the disjoint pairs summing to 15 and then the remaining sums gives exactly the following six divisions. The columns are ordered by their pair sums.

7	9	11	13	15
$(1,6)$	$(2,7)$	$(3,8)$	$(4,9)$	$(5,10)$
$(1,6)$	$(4,5)$	$(2,9)$	$(3,10)$	$(7,8)$
$(2,5)$	$(1,8)$	$(4,7)$	$(3,10)$	$(6,9)$
$(2,5)$	$(3,6)$	$(1,10)$	$(4,9)$	$(7,8)$
$(3,4)$	$(1,8)$	$(2,9)$	$(6,7)$	$(5,10)$
$(3,4)$	$(2,7)$	$(1,10)$	$(5,8)$	$(6,9)$

For example, when the first pair is $(3,4)$, the possible disjoint pairs summing to 15 are $(5,10)$, $(6,9)$ and $(7,8)$. The last leaves no available pair summing to 9; each of the first two gives exactly one row in the table. The other two starting pairs each give the two displayed completions in the same way.

There are therefore seven equally likely divisions satisfying the condition. Exactly one contains $(1,2)$, namely the division with $d=4$. The conditional probability is $\boxed{1/7}$.

19. D

Use the letters in the displayed array. The first and third columns give

$$f = c^2, \qquad e^2 = 225b.$$

The three rows give

$$a = \frac{1+b}{2}, \qquad d = \frac{c+e}{2}, \qquad g = \frac{c^2+225}{2}.$$

The middle column is a geometric progression, so $d^2 = ag$. Substituting the row expressions yields

$$(c+e)^2 = (1+b)(c^2+225).$$

Using $e^2 = 225b$ and cancelling gives

$$2ce = bc^2 + 225 = \frac{c^2e^2}{225} + 225,$$

and therefore

$$(ce - 225)^2 = 0.$$

Thus $ce = 225$ and $b = 225/c^2$.

Since b and c are positive integers, c^2 divides $225 = 3^2 \cdot 5^2$. Hence

$$c = 1, 3, 5 \text{ or } 15.$$

The choice $c = 1$ repeats the upper-left entry, while $c = 15$ gives $b = 1$, also a repeated entry. If $c = 5$, then $b = 9$ and $a = (1+9)/2 = 5 = c$, again violating distinctness.

The only remaining choice is $c = 3$. It gives $e = 75$ and $d = (3+75)/2 = 39$. The full array is

$$\begin{array}{ccc} 1 & 13 & 25 \\ 3 & 39 & 75 \\ 9 & 117 & 225 \end{array}$$

Its entries are distinct positive integers; the row differences are $12, 36, 108$, and each column has common ratio 3. Every condition is met, so $\boxed{d = 39}$.

20. E

No vertex can lie on an axis: rotation through 90° would give a vertex on the y-axis, whereas the only point of the curve on that axis is the origin, which is the square's centre. Choose its first-quadrant vertex (a, b), with $a, b > 0$. The next vertex under a 90° rotation is $(-b, a)$.

Both points lie on the curve, so

$$b = a^3 - 3a, \qquad -a = b^3 - 3b.$$

Write $S = a^2 + b^2$ and $T = a^2b^2$. Multiplying the two displayed equations respectively by a and b, then adding, gives

$$a^4 + b^4 = 3(a^2 + b^2), \qquad S^2 - 2T = 3S.$$

Also, division by the positive numbers a, b and multiplication give

$$(a^2 - 3)(b^2 - 3) = \frac{b}{a}\left(-\frac{a}{b}\right) = -1,$$

so $T = 3S - 10$. Eliminating T yields

$$S^2 - 9S + 20 = 0, \qquad (S - 4)(S - 5) = 0.$$

The square has diagonal length $2\sqrt{S}$, and hence area $2S$. The only candidate areas are 8 and 10.

Both are attainable. For $S = 4$, choose $a, b > 0$ with

$$a^2 = 2 + \sqrt{2}, \qquad b^2 = 2 - \sqrt{2}.$$

Here $ab = \sqrt{2}$, $a^2(a^2-3) = \sqrt{2}$ and $b^2(b^2-3) = -\sqrt{2}$, which verify the two required cubic equations. For $S = 5$, choose

$$a^2 = \frac{5 + \sqrt{5}}{2}, \qquad b^2 = \frac{5 - \sqrt{5}}{2}.$$

Now the same three quantities are respectively $\sqrt{5}, \sqrt{5}, -\sqrt{5}$, again verifying the equations. In each case, the opposite vertices also lie on the curve because $x^3 - 3x$ is an odd function. The four rotated points form the required non-degenerate square.

The sum of the distinct possible areas is therefore $8 + 10 = \boxed{18}$.