1. Let $k \geq 0$. Consider the following statements.

S: For every real number x with $0 \leq x \leq 1$, there exists a real number y such that

$$|y - x| \leq k(1 + x) \qquad \text{and} \qquad |y - x^2| \leq k(1 + x).$$

T: There exists a real number y such that, for every real number x with $0 \leq x \leq 1$,

$$|y - x| \leq k(1 + x) \qquad \text{and} \qquad |y - x^2| \leq k(1 + x).$$

For which values of k is S true and T false?

A $\dfrac{1}{12} \leq k < \dfrac{1}{3}$

B $\dfrac{3 - 2\sqrt{2}}{4} \leq k < \dfrac{1}{3}$

C $\dfrac{3 - 2\sqrt{2}}{2} \leq k < \dfrac{1}{3}$

D $3 - 2\sqrt{2} \leq k < \dfrac{1}{3}$

E $\dfrac{1}{8} \leq k < \dfrac{1}{3}$

F $\dfrac{1}{16} \leq k < \dfrac{1}{3}$

2. A triangle ABC has $AB = 3$, $AC = 4$ and $BC = c$.

Consider the condition

$$c \leq 3\sqrt{2}$$

and the statement

$$AP^2 \geq BP \cdot PC \qquad \text{for every point } P \text{ on the line segment } BC.$$

Which of the following correctly describes the condition in relation to the statement?

A The condition is necessary but not sufficient.

B The condition is sufficient but not necessary.

C The condition is both necessary and sufficient.

D The condition is neither necessary nor sufficient.

3. Each of $a_0, a_1, \ldots, a_6$ is chosen independently, with 1 and -1 equally likely. Define

$$P(x) = x^7 + a_6x^6 + a_5x^5 + \cdots + a_1x + a_0.$$

A student tries to find the probability that P has at least one integer root.

I. If an integer m is a root, then m divides a_0. Thus the only possible integer roots are 1 and -1.

II. $P(1) = 0$ precisely when four of the seven chosen coefficients are -1. Its probability is $\binom{7}{4}/2^7 = 35/128$.

III. The probability that $P(-1) = 0$ is also $35/128$.

IV. Since the coefficients were chosen independently, the events $P(1) = 0$ and $P(-1) = 0$ are independent.

V. The required probability is therefore $2(35/128) - (35/128)^2$.

Which of the following correctly identifies the first error and gives the required probability?

A The first error is in I; the probability is $\dfrac{13}{32}$.

B The first error is in IV; the probability is $\dfrac{13}{32}$.

C The first error is in II; the probability is $\dfrac{35}{64}$.

D The first error is in IV; the probability is $\dfrac{35}{64}$.

E The first error is in V; the probability is $\dfrac{13}{32}$.

F There is no error in the argument.

4. In the coordinate plane, let

$$O = (0,0), \quad A = (1,0), \quad B = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad C = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

A point P lies on the circle

$$x^2 + y^2 - 4x + 2y + 1 = 0.$$

Whenever the three lengths PA, PB and PC can be used as the side lengths of a triangle of positive area, let $K(P)$ be the area of that triangle.

What is the greatest possible value of $K(P)$?

A $\sqrt{15} - 2\sqrt{3}$

B $\dfrac{2\sqrt{3} + \sqrt{15}}{2}$

C $\dfrac{9\sqrt{3} + 4\sqrt{15}}{4}$

D $\sqrt{3} + \sqrt{15}$

E $2\sqrt{3} + \sqrt{15}$

F $4\sqrt{3} + 2\sqrt{15}$

5. A student wishes to prove that there is no real-valued function f, defined for every real number, such that

$$f(x + f(y)) = f(x) + y^2 \qquad \text{for all real } x, y. \tag{$*$}$$

The student assumes that such a function exists and writes the following proof, with one line missing.

Suppose $f(u) = f(v)$. Substituting $y = u$ and $y = v$ into $(*)$ gives $u^2 = v^2$.

Missing line

Hence, if $u \neq v$, then $f(t) = 0$ for every real t, which is incompatible with $(*)$. It follows that f is one-to-one. But taking $x = 0$, first with $y = 1$ and then with $y = -1$, gives

$$f(f(1)) = f(f(-1)).$$

Applying the one-to-one property twice gives $1 = -1$, a contradiction.

Which of the following fills the gap by valid reasoning, without assuming in advance that f is one-to-one?

A For every real t, translating the inputs preserves equality, so $f(u + t) = f(v + t)$. Hence $(u + t)^2 = (v + t)^2$, giving $2(u - v)t = 0$.

B For every real t, $(*)$ gives $f(u + f(t)) = f(v + f(t))$. Cancelling f gives $u + f(t) = v + f(t)$, so $u = v$.

C Since $u^2 = v^2$, either $u = v$ or $u = -v$. In the latter case, $(*)$ with $x = u$ and $y = v$ gives $f(u + f(v)) = f(u) + u^2$, so $f(u) = 0$.

D For every real t, $(*)$ gives $f(u+f(t)) = f(v+f(t))$. Applying the implication already established for equal function values, then subtracting $u^2 = v^2$, gives $2(u - v)f(t) = 0$.

E The equality $f(u) = f(v)$ gives $f(u - v) = 0$. Taking $y = u - v$ in $(*)$ therefore gives $f(x) = f(x) + (u - v)^2$, so $u = v$.

F For every real t, $(*)$ gives $f(u+f(t)) = f(v+f(t))$, and hence $(u+f(t))^2 = (v+f(t))^2$. Taking square roots gives $u + f(t) = v + f(t)$, so $u = v$.

6. The integers $1, 2, \ldots, 27$ are divided into nine groups of three, and each group is replaced by its median. The resulting nine numbers are divided into three groups of three, and again each group is replaced by its median. Finally, the median of the three remaining numbers is taken.

The divisions into groups can be chosen freely at both stages. Which of the following is the complete set of possible final values m?

A Every integer m with $8 \leq m \leq 20$.

B Every integer m with $7 \leq m \leq 21$.

C Every integer m with $9 \leq m \leq 19$.

D Every integer m with $10 \leq m \leq 18$.

E Every integer m with $13 \leq m \leq 15$.

F $m = 14$ only.

7. Eight distinct real numbers have the following property:

From every choice of three of the numbers, it is possible to select two whose sum is rational.

What is the smallest possible number of rational numbers among the eight?

A 0

B 4

C 6

D 5

E 7

F 8

8. Each of two bags contains red and blue counters, with at least one counter of each colour in each bag. The first bag contains more red counters than the second bag.

For a real number p with $0 < p < 1$, an experiment consists of choosing the first bag with probability p, choosing the second bag otherwise, and then drawing two counters successively from the chosen bag without replacement. Each remaining counter in that bag is equally likely to be drawn.

Let $U(p)$ be the probability that the first counter is red. Let $R(p)$ be the probability that the second counter is red, given that the first is red, and let $B(p)$ be the probability that the second counter is blue, given that the first is blue.

It is given that

$$R\left(\frac{1}{3}\right) = R\left(\frac{2}{3}\right).$$

Which statement must be true as p increases with $0 < p < 1$?

A $U(p)$ and $B(p)$ are both strictly increasing.

B $U(p)$ and $B(p)$ are both strictly decreasing.

C $U(p)$ is strictly increasing and $B(p)$ is strictly decreasing.

D $U(p)$ is constant and $B(p)$ is strictly increasing.

E $U(p)$ is strictly decreasing and $B(p)$ is strictly increasing.

F $U(p)$ is strictly decreasing and $B(p)$ is constant.

9. Five boxes are arranged in a circle. In clockwise order, their initial numbers of counters are

$$0, \quad 1, \quad 0, \quad 2, \quad 4.$$

In one move, two adjacent boxes are chosen and one counter is added to each of them. The first and fifth boxes in the displayed order are adjacent. Counters are never removed.

What is the complete set of integers n for which a finite sequence of moves can leave exactly n counters in each box?

A All integers $n \geq 4$

B All odd integers $n \geq 5$

C All odd integers $n \geq 7$

D All integers $n \geq 7$

E All even integers $n \geq 8$

F All odd integers $n \geq 9$

10. A polynomial $P(x)$ with real coefficients has degree 11, and $P(1) \neq 0$. Every real root is simple: whenever $P(r) = 0$, the expression $(x - r)^2$ is not a factor of $P(x)$.

It is also given that, for every real root r of $P(x)$,

$$P\left(\frac{1}{1-r}\right) = 0.$$

Which is the complete list of possible numbers of distinct real roots of $P(x)$?

A 3 only

B 3 and 6 only

C 6 and 9 only

D 3 and 9 only

E 3, 6 and 9 only

F 1, 3, 5, 7, 9 and 11

11. A right-angled triangle has positive integer side lengths $a < b < c$ and area 60. Let h be the perpendicular distance from the right-angled vertex to the hypotenuse.

What is the largest possible value of

$$\frac{h}{b-a}\,?$$

A $\dfrac{60}{119}$

B $\dfrac{120}{289}$

C $\dfrac{120}{119}$

D $\dfrac{120}{17}$

E $\dfrac{30}{\sqrt{61}}$

12. The polynomial

$$f(x) = x^2 + ax + b$$

satisfies

$$\int_0^1 f(x)\,\mathrm{d}x = 0, \qquad \int_0^1 xf(x)\,\mathrm{d}x = 0.$$

What is the total area between the curve $y = f(x)$ and the x-axis for $0 \leq x \leq 1$?

A 0

B $\dfrac{\sqrt{3}}{108}$

C $\dfrac{\sqrt{3}}{54}$

D $\dfrac{\sqrt{3}}{27}$

E $\dfrac{2\sqrt{3}}{27}$

13. Let a and b be real constants. Consider the condition

$$a^2 \leq 4b$$

and the following statement:

For every real angle θ, if

$$\sin^2\theta + a\sin\theta\cos\theta + b\cos^2\theta = 0,$$

then $\sin^2\theta = \cos^2\theta$.

Which of the following correctly describes the condition in relation to the statement?

A The condition is necessary but not sufficient.

B The condition is sufficient but not necessary.

C The condition is both necessary and sufficient.

D The condition is neither necessary nor sufficient.

14. A geometric progression has terms

$$a_n = ar^{n-1} \qquad (n = 1, 2, 3, \ldots),$$

where $a > 0$ and $r > 0$. It is known that a_2, a_5 and a_9 are integers.

Consider the condition $r \geq 1$ and the statement that infinitely many terms of the progression are integers.

Which of the following correctly describes the condition in relation to the statement?

A The condition is necessary but not sufficient.

B The condition is sufficient but not necessary.

C The condition is both necessary and sufficient.

D The condition is neither necessary nor sufficient.

15. The vertices of a convex quadrilateral are A, B, C, D, in that order. A point P in the plane is called balanced if

$$PA^2 + PC^2 = PB^2 + PD^2.$$

Consider the condition that every point on the diagonal AC is balanced, and the statement that $ABCD$ is a rectangle.

Which of the following correctly describes the condition in relation to the statement?

A quadrilateral is convex if each of its interior angles is less than $180°$.

A The condition is necessary but not sufficient.

B The condition is sufficient but not necessary.

C The condition is both necessary and sufficient.

D The condition is neither necessary nor sufficient.

16. Consider the inequality

$$a^n < 1 + n(a-1).$$

Which of the following statements are true?

I. There exists a positive integer n for which the inequality holds for every real number $a < -1$.

II. For every real number $a < -1$, there exists a positive integer n for which the inequality holds.

III. For every real number $a < -1$, there exists a positive integer N for which the inequality holds for every integer $n \geq N$.

A None of them

B I and II only

C II and III only

D II only

E I, II and III

17. How many composite positive integers N have the following property?

For every pair of integers a, b with $a > 1$ and $b > 1$, if $ab = N$, then $a + b$ is a factor of N.

A 0

B 1

C 2

D 3

E Infinitely many

18. For real numbers x and y, define

$$s_n = x^n + y^n$$

for each positive integer n.

Consider the condition that s_1, s_2 and s_4 are integers, and the statement that s_n is an integer for every positive integer n.

Which of the following correctly describes the condition in relation to the statement?

A The condition is necessary but not sufficient.

B The condition is sufficient but not necessary.

C The condition is both necessary and sufficient.

D The condition is neither necessary nor sufficient.

19. A sequence of real numbers is defined by

$$0 < x_1 < 1, \qquad x_{n+1} = |2x_n - 1| \quad (n \geq 1).$$

Consider the condition that $x_n = x_1$ for some integer $n > 1$, and the statement that x_1 is rational. Which of the following correctly describes the condition in relation to the statement?

A The condition is necessary but not sufficient.

B The condition is sufficient but not necessary.

C The condition is both necessary and sufficient.

D The condition is neither necessary nor sufficient.

20. A function f maps every positive integer to a positive integer and satisfies

$$f(f(n)) = n + 4$$

for every positive integer n.

How many different functions f satisfy this condition?

A 2

B 3

C 6

D 12

E 24

F Infinitely many