**1. C**

For a fixed $x$ with $0 \le x \le 1$, the two target values are $x^2$ and $x$, with $x^2 \le x$. A number $y$ can be within $k(1+x)$ of both precisely when

$$x - x^2 \le 2k(1+x).$$

Necessity follows by adding the two distance bounds; for sufficiency take $y = (x + x^2)/2$.

Put $r = \sqrt{2} - 1$ and $h = 3 - 2\sqrt{2} = r^2$. The identity

$$h(1+x) - x(1-x) = (x-r)^2$$

shows that

$$\frac{x(1-x)}{1+x} \le h$$

throughout $0 \le x \le 1$, with equality at $x = r$, which lies strictly between 0 and 1. Consequently S is true precisely when

$$k \ge \frac{3-2\sqrt{2}}{2}.$$

For T, applying its inequalities at $x = 0$ and $x = 1$ gives $|y| \le k$ and $|y-1| \le 2k$. Adding the distances yields $1 \le 3k$, so $k \ge 1/3$ is necessary.

It is also sufficient. At $k = 1/3$, choose the single value $y = 1/3$. For $0 \le x \le 1$,

$$\frac{1}{3} - x \le \frac{1+x}{3}, \qquad x - \frac{1}{3} \le \frac{1+x}{3},$$

and

$$\frac{1}{3} - x^2 \le \frac{1+x}{3}, \qquad x^2 - \frac{1}{3} \le \frac{1+x}{3}.$$

The last inequality is equivalent to $(3x+2)(x-1) \le 0$; the others follow directly from $0 \le x \le 1$. Increasing $k$ preserves the bounds. Thus T is true precisely when $k \ge 1/3$.

S is therefore true and T false precisely when

$$\boxed{\frac{3-2\sqrt{2}}{2} \le k < \frac{1}{3}}.$$

The lower endpoint is included because the inequalities are non-strict, whereas the upper endpoint is excluded because T is already true there.

**2. B**

Write $BP = sc$, so $PC = (1-s)c$ and $0 \le s \le 1$. Take coordinates $B = (0,0)$, $C = (c,0)$ and $A = (u,v)$. The given sides give

$$u^2 + v^2 = 9, \qquad (u-c)^2 + v^2 = 16,$$

so $2uc = c^2 - 7$. Since $P = (sc, 0)$,

$$AP^2 = (u - sc)^2 + v^2 = 9 + 7s - c^2 s(1-s).$$

It follows that

$$\begin{aligned} AP^2 - BP \cdot PC &= 9 + 7s - 2c^2 s(1-s) \\ &= (7s-3)^2 + (49 - 2c^2)s(1-s). \end{aligned}$$

If $c \le 3\sqrt{2}$, then $c^2 \le 18 < 49/2$, so both terms on the last line are non-negative for every allowed $s$. The condition is sufficient.

It is not necessary. Take $c = 7/\sqrt{2}$. This is a valid non-degenerate triangle because $1 < c < 7$, and the identity becomes

$$AP^2 - BP \cdot PC = (7s-3)^2 \ge 0$$

for every point on $BC$. Thus the statement holds, although $c > 3\sqrt{2}$.

In fact, the sharp bound for the statement is $c \le 7/\sqrt{2}$: the identity proves sufficiency, and choosing $s = 3/7$ proves necessity.

The smaller bound in the question is therefore $\boxed{\text{sufficient but not necessary}}$.

**3. B**

Lines I–III are correct. For I, an integer root $m$ is nonzero because $a_0 \neq 0$, and $P(m) = 0$ gives

$$a_0 = -m(m^6 + a_6 m^5 + \cdots + a_1),$$

so $m$ divides $a_0 = \pm 1$. For III, after substituting $x = -1$, the seven random signs still range independently over 1 and $-1$; their sum must be 1, giving $\binom{7}{4} = 35$ possibilities.

Line IV is the first error: independence of the seven coefficients does not imply independence of two events determined by those same coefficients.

To count their intersection, write

$$E = a_6 + a_4 + a_2 + a_0, \qquad O = a_5 + a_3 + a_1.$$

Then

$$P(1) = 1 + E + O, \qquad P(-1) = -1 + E - O.$$

Both vanish precisely when $E = 0$ and $O = -1$. There are $\binom{4}{2} = 6$ ways to choose the four even-indexed signs and $\binom{3}{2} = 3$ ways to choose the three odd-indexed signs. These are disjoint groups of independent choices, giving 18 coefficient lists out of $2^7 = 128$.

Thus the number of lists giving at least one integer root is $35 + 35 - 18 = 52$, and the required probability is

$$\boxed{\frac{52}{128} = \frac{13}{32}}.$$

For comparison, the actual probability that both events occur is $18/128 = 9/64$, which differs from $(35/128)^2$. Line V would be a valid calculation under the incorrect independence assertion; the earlier error is IV.

**4. E**

Write $P = (x, y)$ and $t = x^2 + y^2 = OP^2$. Put $a = PA$, $b = PB$ and $c = PC$. Then

$$a^2 = t - 2x + 1, \qquad b^2 = t + x + 1 - \sqrt{3}\,y, \qquad c^2 = t + x + 1 + \sqrt{3}\,y.$$

Hence

$$\begin{aligned} 4b^2c^2 - (b^2 + c^2 - a^2)^2 &= 4\big((t + x + 1)^2 - 3y^2\big) - (t + 4x + 1)^2 \\ &= 3(t + 1)^2 - 12(x^2 + y^2) = 3(t - 1)^2. \end{aligned} \tag{1}$$

When the distances form a triangle, let $\alpha$ be its angle opposite the side of length $a$. The cosine rule and area formula give

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}, \qquad K(P) = \frac{1}{2}bc\sin\alpha.$$

Using (1),

$$16K(P)^2 = 3(t - 1)^2, \qquad K(P) = \frac{\sqrt{3}}{4}|t - 1|.$$

Completing the square in the circle equation gives

$$(x - 2)^2 + (y + 1)^2 = 4.$$

Its centre is at distance $\sqrt{5}$ from $O$, and its radius is 2. Thus

$$\sqrt{5} - 2 \le OP \le \sqrt{5} + 2, \qquad 9 - 4\sqrt{5} \le t \le 9 + 4\sqrt{5},$$

with both endpoints attained on the line through $O$ and the centre. Since $9 - 4\sqrt{5} < 1$, the largest possible value of $|t - 1|$ is

$$\max\{4\sqrt{5} - 8,\ 8 + 4\sqrt{5}\} = 8 + 4\sqrt{5}.$$

The farthest point has $t = 9 + 4\sqrt{5} \neq 1$. At that point $a, b, c > 0$, because $A, B, C$ all have distance 1 from $O$. Equation (1) is strictly positive, so

$$-2bc < b^2 + c^2 - a^2 < 2bc, \quad \text{giving} \quad |b - c| < a < b + c.$$

These strict triangle inequalities show that the maximizing point is allowed. Therefore

$$\boxed{K_{\max} = \frac{\sqrt{3}}{4}(8 + 4\sqrt{5}) = 2\sqrt{3} + \sqrt{15}}.$$

**5. D**

The missing line is D.

First, whenever $f(a) = f(b)$, evaluating $(*)$ with the same $x$ and with $y = a, b$ gives

$$f(x + f(a)) = f(x) + a^2, \qquad f(x + f(b)) = f(x) + b^2.$$

The left sides are equal, so $a^2 = b^2$. This implication does not assume that $f$ is one-to-one.

Now suppose $f(u) = f(v)$. For any real $t$, $(*)$ gives

$$f(u + f(t)) = f(u) + t^2 = f(v) + t^2 = f(v + f(t)).$$

Apply the implication just proved to the two inputs $u + f(t)$ and $v + f(t)$. It gives

$$(u + f(t))^2 = (v + f(t))^2.$$

Expanding and using $u^2 = v^2$ leaves

$$2(u - v)f(t) = 0 \qquad \text{for every real } t.$$

If $u \neq v$, division by the nonzero constant $2(u - v)$ forces $f(t) = 0$ for all $t$. But the zero function fails $(*)$ at $y = 1$. Therefore $f(u) = f(v)$ forces $u = v$; $f$ is one-to-one.

Taking $x = 0$ and $y = \pm 1$ gives $f(f(1)) = f(f(-1)) = f(0) + 1$. One-to-one-ness first gives $f(1) = f(-1)$, and then $1 = -1$. This completes the contradiction.

A assumes invariance under an arbitrary translation without establishing it. B uses the one-to-one property while it is still being proved. C draws $f(u) = 0$ from an equation that does not imply it. E assumes that equal function values imply $f(u - v) = 0$, which would need an additional argument, such as an established additive law. F replaces equality of squares by equality of the numbers themselves, ignoring their possible opposite signs. None of these is a valid completion as written.

**6. A**

A median of three numbers has at least two of those numbers no greater than it. At the first stage, every selected number therefore has at least two original numbers no greater than it. At the second stage, each selected number has at least four original numbers no greater than it, because its two contributing groups are disjoint. At the final stage, at least eight original numbers are no greater than $m$. Hence $m \geq 8$.

Applying the same argument with "no less than" gives at least eight original numbers no less than $m$, so $28 - m \geq 8$ and $m \leq 20$.

To prove that every integer between these bounds is possible, choose any $m$ with $8 \leq m \leq 20$. Select seven numbers below $m$, denoted $L_1, \ldots, L_7$, and seven above $m$, denoted $H_1, \ldots, H_7$. This is possible by the bounds. Call the remaining twelve numbers $R_1, \ldots, R_{12}$; they may lie on either side of $m$.

Use the following nine triples, and at the second stage group their medians in the three rows shown:

$$\begin{array}{ccc} (L_1, m, H_1) & (L_2, L_3, R_1) & (H_2, H_3, R_2) \\ (L_4, L_5, R_3) & (L_6, L_7, R_4) & (R_5, R_6, R_7) \\ (H_4, H_5, R_8) & (H_6, H_7, R_9) & (R_{10}, R_{11}, R_{12}). \end{array}$$

In the first row the three medians are $m$, a number below $m$ and a number above $m$, so its next median is $m$. In the second row at least two medians are below $m$, so its next median is below $m$. In the third row at least two medians are above $m$, so its next median is above $m$.

The final median is therefore exactly $m$. All original numbers are used once. Thus the complete set is every integer $m$ with $\boxed{8 \leq m \leq 20}$.

**7. C**

First show that at least one of the eight numbers is rational. If all are rational there is nothing to prove; otherwise choose an irrational number $a$ from the list.

At most two of the other numbers can have a rational sum with $a$. Indeed, if three of them, say $b, c, d$, did, then every pair among $b, c, d$ would have an irrational sum: for example,

$$b + c = (a + b) + (a + c) - 2a$$

is irrational. This would violate the given property for the triple $b, c, d$.

Consequently at least five of the other numbers have an irrational sum with $a$. The sum of any two of these five must be rational, by applying the given property to those two numbers together with $a$. Choose three of the five, say $u, v, w$. Their three pair sums are rational, so

$$2u = (u + v) + (u + w) - (v + w)$$

is rational. Thus the list contains a rational number, say $r$.

Now consider any two irrational numbers $s, t$ in the list. In the triple $r, s, t$, neither $r + s$ nor $r + t$ is rational, so $s + t$ must be rational. If the list contained three irrational numbers, all three of their pair sums would therefore be rational, and the same displayed identity would make each of those three numbers rational, a contradiction.

There are at most two irrational numbers, so at least six entries are rational. This bound is attained by

$$1, 2, 3, 4, 5, 6, \sqrt{2}, -\sqrt{2}.$$

Any triple either contains two integers, whose sum is rational, or contains both $\sqrt{2}$ and $-\sqrt{2}$, whose sum is 0. The eight entries are distinct. The smallest possible number of rational entries is therefore $\boxed{6}$.

**8. E**

Write $r_i, b_i$ and $n_i = r_i + b_i$ for the red, blue and total counts in bag $i$. The counts $r_i, b_i$ are positive, and $r_1 > r_2$.

Put

$$u_i = \frac{r_i}{n_i} > 0, \qquad v_i = \frac{r_i - 1}{n_i - 1}.$$

Conditioning on a red first counter gives

$$R(p) = \frac{pu_1v_1 + (1-p)u_2v_2}{pu_1 + (1-p)u_2}.$$

The given equality is therefore

$$\frac{u_1v_1 + 2u_2v_2}{u_1 + 2u_2} = \frac{2u_1v_1 + u_2v_2}{2u_1 + u_2}.$$

Both denominators are positive. Cross-multiplication and cancellation give $3u_1u_2(v_2 - v_1) = 0$, so $v_1 = v_2 = c$, say.

Since $r_1 > r_2 \geq 1$, we have $r_1 \geq 2$ and hence $c > 0$. Each bag also contains a blue counter, so $c < 1$. From

$$r_i - 1 = c(n_i - 1)$$

and $r_1 > r_2$, it follows that $n_1 > n_2$. Moreover,

$$u_i = \frac{r_i}{n_i} = c + \frac{1-c}{n_i}.$$

Thus $u_1 < u_2$. Since

$$U(p) = pu_1 + (1-p)u_2,$$

$U(p)$ is strictly decreasing. In particular, having more red counters does not mean that the first bag has the greater proportion of red counters.

For blue, put

$$t_i = \frac{b_i}{n_i} > 0, \qquad w_i = \frac{b_i - 1}{n_i - 1} = 1 - c - \frac{1}{n_i - 1}.$$

As $n_1 > n_2$, we have $w_1 > w_2$. Also,

$$B(p) = \frac{pt_1w_1 + (1-p)t_2w_2}{pt_1 + (1-p)t_2}.$$

For any $0 < p < q < 1$, cross-multiplication gives

$$B(q) - B(p) = \frac{(q-p)t_1t_2(w_1 - w_2)}{(qt_1 + (1-q)t_2)(pt_1 + (1-p)t_2)} > 0.$$

All factors in the denominator are positive. Hence $B(p)$ is strictly increasing, giving option E.

The premises are consistent: bags with $(r_1, b_1) = (3, 4)$ and $(r_2, b_2) = (2, 2)$ have $v_1 = v_2 = 1/3$ and $r_1 > r_2$.

**9. C**

Number the boxes $1, 2, 3, 4, 5$ in the displayed order, and let $M$ be the number of moves. Initially there are 7 counters, and each move adds 2, so

$$5n = 7 + 2M.$$

It follows that $n$ is odd.

Boxes 1 and 3 are initially empty and are not adjacent. Each needs $n$ counters, and no move can add to both of them. At least $2n$ moves are therefore required. Hence

$$\frac{5n - 7}{2} = M \geq 2n,$$

which gives $n \geq 7$.

Both restrictions can be attained. To reach $n = 7$, make respectively

$$4, \quad 2, \quad 5, \quad 0, \quad 3$$

moves on the adjacent pairs

$$(1, 2),\ (2, 3),\ (3, 4),\ (4, 5),\ (5, 1).$$

The final counts are

$$0 + 4 + 3 = 7, \quad 1 + 4 + 2 = 7, \quad 0 + 2 + 5 = 7, \quad 2 + 5 + 0 = 7, \quad 4 + 0 + 3 = 7.$$

The order of these moves does not matter, since counters are only added.

Making one further move on each of the five adjacent pairs adds 2 to every box. Repeating this procedure reaches every odd $n \geq 7$. Thus the stated parity and lower bound are both necessary and sufficient.

**10. D**

The number 1 is not a root. Nor can 0 be a root, since the given implication would then make 1 a root.

For a root $r$, successively applying the given implication produces

$$r, \qquad \frac{1}{1-r}, \qquad \frac{r-1}{r}, \qquad r.$$

All denominators are nonzero. These first three numbers are distinct. A fixed point would satisfy

$$r = \frac{1}{1-r}, \qquad r^2 - r + 1 = 0,$$

which has no real solution, since $(r - 1/2)^2 + 3/4 > 0$. If the first and third numbers coincided, applying the transformation once more would give a fixed point as well.

Thus the real roots fall into disjoint groups of three: beginning at any one member gives the same three members. Their number $N$ is a positive multiple of 3 and is at most 11.

Also, $N$ must be odd. Indeed, divide $P$ by the product of its $N$ distinct real linear factors. Because the real roots are simple and all of them have been included, the resulting polynomial has no real root. A real polynomial of odd degree has opposite signs for sufficiently large positive and negative inputs and therefore has a real root. The resulting degree $11 - N$ must consequently be even. This argument does not require complex roots.

The only possibilities left are $N = 3$ and $N = 9$. Both occur. Define

$$A_s(x) = (x - s)\left(x - \frac{1}{1-s}\right)\left(x - \frac{s-1}{s}\right), \qquad s > 1.$$

Then

$$A_2(x)(x^2 + 1)^4$$

has degree 11 and exactly three real roots, all simple. For nine real roots, use

$$A_2(x)A_3(x)A_4(x)(x^2 + 1).$$

Its three groups are

$$2, -1, \frac{1}{2}; \qquad 3, -\frac{1}{2}, \frac{2}{3}; \qquad 4, -\frac{1}{3}, \frac{3}{4}.$$

These nine numbers are distinct, each group is preserved by the required transformation, and none is 1. The extra factors $x^2 + 1$ introduce no real roots. Hence both constructions satisfy every condition.

**11. C**

The two shorter sides are the perpendicular sides, so the area condition and the altitude give

$$\frac{ab}{2} = 60 = \frac{ch}{2}, \qquad ab = 120, \qquad h = \frac{120}{c}.$$

Put $d = b - a > 0$. Pythagoras' theorem gives

$$c^2 = a^2 + b^2 = (b-a)^2 + 2ab = d^2 + 240.$$

Consequently

$$\frac{h}{b-a} = \frac{120}{d\sqrt{d^2+240}}.$$

Both positive factors in the denominator increase as $d$ increases. The ratio is therefore greatest when an allowed triangle has the smallest possible $d$.

Since $ab = 120$ and $a < b$, the integer $a$ is a divisor of 120 smaller than $\sqrt{120}$. The largest two such divisors are 10 and 8. The pair $(a, b) = (10, 12)$ gives $c^2 = 244$, strictly between $15^2$ and $16^2$, so it is not allowed. The pair $(8, 15)$ gives $c = 17$, an integer, and $d = 7$.

For every smaller possibility $a < 8$, we have $b = 120/a > 15$, so $b - a > 7$. Thus $(8, 15, 17)$ attains the smallest allowed difference and gives

$$\boxed{\frac{h}{b-a} = \frac{120}{17 \cdot 7} = \frac{120}{119}}.$$

**12. D**

Integrating the two given expressions gives

$$\frac{1}{3}+\frac{a}{2}+b=0, \qquad \frac{1}{4}+\frac{a}{3}+\frac{b}{2}=0.$$

Subtracting half the first equation from the second gives $1/12 + a/12 = 0$, so $a = -1$ and then $b = 1/6$. Thus

$$f(x)=x^2-x+\frac{1}{6}=\left(x-\frac{1}{2}\right)^2-\frac{1}{12}.$$

Its two zeros are

$$\alpha=\frac{1}{2}-\frac{\sqrt{3}}{6}, \qquad \beta=\frac{1}{2}+\frac{\sqrt{3}}{6},$$

both strictly between 0 and 1. The function is negative between the zeros and positive on the remaining two parts of the interval.

The total signed integral is zero, so the positive area equals the magnitude of the negative integral. Therefore the total unsigned area is

$$-2\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x.$$

Put $r = \sqrt{3}/6$, so $r^2 = 1/12$. An antiderivative of $f$ is

$$F(x)=\frac{(x-1/2)^3}{3}-\frac{x-1/2}{12}.$$

At $x = 1/2 + r$ and $x = 1/2 - r$, this gives

$$\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x=\frac{2r^3}{3}-\frac{r}{6}=\frac{r}{18}-\frac{r}{6}=-\frac{r}{9}=-\frac{\sqrt{3}}{54}.$$

The required area is consequently

$$\boxed{\frac{\sqrt{3}}{27}}.$$

The given zero integral is a signed area, not the total area requested.

**13. D**

The condition is not sufficient. Take $a = b = 0$, so $a^2 \leq 4b$ holds. At $\theta = 0$, the premise becomes $\sin^2 0 = 0$, which is true, but the conclusion would be $0 = 1$, which is false. Thus the universal statement fails.

The condition is not necessary either. Take $a = 0$ and $b = -1$. The premise is then

$$\sin^2 \theta - \cos^2 \theta = 0,$$

which is precisely the conclusion. Hence the statement is true for every angle, although $a^2 \leq 4b$ is false since $0 \nleq -4$.

The discriminant explains the trap. When $\cos \theta = 0$, the premise is false. Otherwise, dividing by $\cos^2 \theta$ and putting $t = \tan \theta$ reduces the premise to

$$t^2 + at + b = 0,$$

while the conclusion requires $t = 1$ or $t = -1$. The strict condition $a^2 < 4b$ makes the implication true because the quadratic has no real roots. At equality, however, its repeated root need not be 1 or $-1$, as the first counterexample shows. Conversely, the quadratic $t^2 - 1$ has two real roots that are both allowed.

Therefore the stated condition is $\boxed{\text{neither necessary nor sufficient}}$.

**14. A**

The specified integer terms force $r$ to be rational, because

$$\frac{a_9 a_2}{a_5^2} = \frac{(ar^8)(ar)}{(ar^4)^2} = r.$$

Write $r = u/v$ in lowest terms, where $u, v$ are positive integers. Let $N = a_2$, a positive integer. For $n \geq 2$,

$$a_n = N\frac{u^{n-2}}{v^{n-2}}.$$

If $a_n$ is an integer, coprimality implies that $v^{n-2}$ divides $N$. If $v > 1$, these powers eventually exceed $N$, so only finitely many terms can be integers. Thus infinitely many integer terms require $v = 1$, making $r$ a positive integer and in particular $r \geq 1$. The condition is necessary.

It is not sufficient. Take

$$a = 2^9, \qquad r = \frac{3}{2}.$$

Then

$$a_n = 2^{10-n}3^{n-1}.$$

Exactly the first ten terms are integers: after $n = 10$, the denominator contains a positive power of 2 and the numerator is odd. The terms $a_2, a_5, a_9$ are all among those ten integers, and $r \geq 1$, but there are not infinitely many integer terms.

For comparison, if $r$ is a positive integer, every term from the integer $a_2$ onwards is an integer. The exact criterion is therefore stronger than $r \geq 1$. The condition in the question is $\boxed{\text{necessary but not sufficient}}$.

**15. A**

First suppose $ABCD$ is a rectangle. Its diagonals have the same midpoint and the same length. Taking their common midpoint as the origin, write

$$A = (h, k), \quad C = (-h, -k), \quad B = (p, q), \quad D = (-p, -q),$$

where $h^2 + k^2 = p^2 + q^2$. For any point $P = (u, v)$, expanding the squared distances gives

$$PA^2 + PC^2 = 2(u^2 + v^2) + 2(h^2 + k^2),$$
$$PB^2 + PD^2 = 2(u^2 + v^2) + 2(p^2 + q^2).$$

Thus every point of the plane is balanced, including every point on $AC$. The condition is necessary.

It is not sufficient. Take

$$A = (-5, 0), \quad B = (0, 1), \quad C = (5, 0), \quad D = (0, -7).$$

These vertices in the displayed order form a convex quadrilateral: the two triangles on base $AC$ lie on opposite sides of $AC$, and both remaining vertices project strictly inside $AC$. Direct expansion for $P = (u, v)$ gives

$$PA^2 + PC^2 = 2u^2 + 2v^2 + 50, \qquad PB^2 + PD^2 = 2u^2 + 2v^2 + 12v + 50.$$

Every point of $AC$ has $v = 0$ and is balanced. However, the midpoints of $AC$ and $BD$ are $(0, 0)$ and $(0, -3)$, respectively. The quadrilateral is therefore not a parallelogram and cannot be a rectangle.

Hence the condition is $\boxed{\text{necessary but not sufficient}}$.

**16. D**

For every $a < -1$, the right-hand side is negative:

$$1 + n(a-1) < 1 - 2n < 0.$$

If $n$ is even, then $a^n > 0$, so the inequality cannot hold. There are arbitrarily large even integers, and therefore III is false.

To test I, a single positive integer $n$ would have to work for every $a < -1$. No even $n$ works, and $n = 1$ gives equality. For any fixed odd integer $n \geq 3$, take

$$a = -2^{1/n}.$$

Then $a < -1$ and $a^n = -2$, whereas

$$1 + n(a-1) = 1 - n(1 + 2^{1/n}) < 1 - 2n \leq -5.$$

Thus this choice of $a$ does not satisfy the inequality. Every possible fixed $n$ has been excluded, so I is false.

For II, fix any $a < -1$ and write $a = -1 - t$, where $t > 0$. Choose an odd integer $n > 1$ such that $nt^2 > 4$. Such an odd integer exists for every fixed positive $t$.

All terms of the binomial expansion of $(1+t)^n$ are positive. Keeping the first three terms gives

$$(1+t)^n \geq 1 + nt + \frac{n(n-1)}{2}t^2 > 1 + nt + 2(n-1) = n(2+t) - 1.$$

Since $n$ is odd, it follows that

$$a^n = -(1+t)^n < 1 - n(2+t) = 1 + n(a-1).$$

The choice of $n$ may depend on $a$, exactly as II allows. Hence II is true.

Only II is true, giving D.

**17. B**

The number $N = 4$ has the property: its only factorisation with both factors greater than 1 is $4 = 2 \cdot 2$, and $2 + 2$ divides 4.

Suppose a composite $N$ has the property, and let $p$ be its smallest prime factor. Write $N = pb$, where $b > 1$. The property says that $p + b$ divides $pb$. It must therefore also divide

$$p(p + b) - pb = p^2.$$

The positive divisors of $p^2$ are $1, p, p^2$. Since $p + b > p$, we must have

$$p + b = p^2, \qquad b = p(p - 1), \qquad N = p^2(p - 1).$$

If $p > 2$, then $p-1 > 1$ has a prime factor smaller than $p$. That prime would divide $N$, contradicting the choice of $p$ as the smallest prime factor of $N$.

Therefore $p = 2$, which forces $N = 4$. There is exactly one composite positive integer with the property, giving B.

**18. C**

The condition is necessary: if every positive-indexed power sum is an integer, then in particular $s_1, s_2, s_4$ are integers.

For sufficiency, suppose $s_1, s_2, s_4$ are integers. Put

$$u = x + y = s_1, \qquad k = s_1^2 - s_2 = 2xy.$$

Both $u$ and $k$ are integers. Also

$$s_4 = (x^2 + y^2)^2 - 2x^2y^2 = s_2^2 - \frac{k^2}{2}.$$

Since $s_2$ and $s_4$ are integers, $k^2$ is even. An odd integer has an odd square, so $k$ is even. Hence $v = xy = k/2$ is an integer.

For every positive integer $n$,

$$s_{n+2} = u\, s_{n+1} - v\, s_n.$$

Indeed, expanding $(x + y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n)$ leaves $x^{n+2} + y^{n+2}$.

The coefficients $u, v$ and the starting values $s_1, s_2$ are integers. Repeated use of the recurrence therefore makes $s_3, s_4, s_5, \ldots$ integers. No separate assumption about $s_3$ is needed.

The condition is thus $\boxed{\text{both necessary and sufficient}}$.

**19. B**

Suppose the condition holds, and write $k = n - 1 > 0$ for the number of steps before the return. At each step, resolving the absolute value gives either

$$x_{j+1} = 2x_j - 1 \qquad \text{or} \qquad x_{j+1} = 1 - 2x_j.$$

Following the actual choices through the $k$ steps gives

$$x_n = \varepsilon 2^k x_1 + c,$$

where $\varepsilon$ is 1 or $-1$ and $c$ is an integer. This follows by repeated substitution: each step multiplies the coefficient of $x_1$ by 2 or $-2$ and leaves an integer constant.

Since $x_n = x_1$,

$$(1 - \varepsilon 2^k)x_1 = c.$$

The coefficient is a nonzero integer because $k \geq 1$. Hence $x_1$ is rational, proving sufficiency.

For non-necessity, take $x_1 = 2/3$. Then

$$x_2 = \frac{1}{3}, \qquad x_3 = \frac{1}{3},$$

and every later term is $1/3$. The first term is rational and lies strictly between 0 and 1, but it never returns. Eventual repetition does not imply a return to the initial term.

The condition is therefore $\boxed{\text{sufficient but not necessary}}$.

**20. D**

Applying $f$ to the given identity, and also applying the identity with input $f(n)$, gives

$$f(n+4) = f(f(f(n))) = f(n) + 4.$$

Thus, for every $r$ among $1, 2, 3, 4$ and every integer $j \geq 0$,

$$f(r+4j) = f(r) + 4j. \tag{1}$$

The entire function is determined by its first four values.

For one of these inputs $r$, write

$$f(r) = s + 4k,$$

where $s$ is one of $1, 2, 3, 4$ and $k$ is a nonnegative integer. Using (1),

$$r + 4 = f(f(r)) = f(s) + 4k,$$

so

$$f(s) = r + 4(1-k).$$

Positivity of $f(s)$, together with $r \leq 4$, forces $k = 0$ or $k = 1$. Also $s \neq r$: otherwise the two expressions for $f(r)$ would imply $k = 1 - k$, impossible for an integer $k$.

Consequently the four inputs $1, 2, 3, 4$ are paired. For each pair $r, s$, exactly one of the following two arrangements holds:

$$f(r) = s, \quad f(s) = r + 4,$$

or

$$f(r) = s + 4, \quad f(s) = r.$$

There are three ways to split four inputs into two pairs, because the partner of 1 may be 2, 3 or 4, after which the other pair is determined. For each pairing the two arrangements may be chosen independently for the two pairs, giving $2 \cdot 2 = 4$ choices.

Conversely, each of these choices defines positive first values, and extension by (1) gives a function on all positive integers. For either arrangement of every pair, direct substitution verifies $f(f(r+4j)) = r + 4j + 4$. Thus every counted function works, and different choices give different functions.

There are $3 \cdot 4 = 12$ functions, giving D.