1. A polynomial $P(x)$ of degree 3 has three distinct real roots a, b and c. It is given that

$$P(a+b) = P(b+c) = P(c+a) = 4$$

and that $P(1) = -1$.

What is the value of $P(-1)$?

A -5

B -3

C 1

D 3

E 5

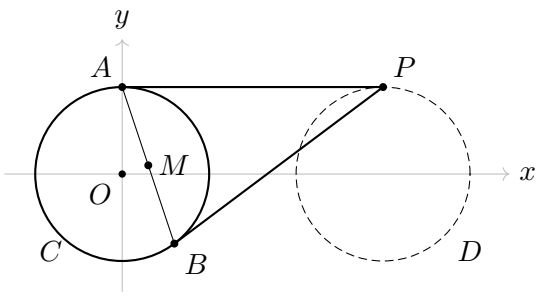
F 9

2. The circles C and D have equations

$$C: \ x^2 + y^2 = 1, \qquad D: \ (x-3)^2 + y^2 = 1.$$

A point P moves once around D. The two tangents from P to C meet C at A and B, and M is the midpoint of AB.

What is the area enclosed by the curve traced by M?



One possible position of P is shown.

A $\dfrac{\pi}{256}$

B $\dfrac{\pi}{64}$

C $\dfrac{\pi}{16}$

D $\dfrac{\pi}{8}$

E $\dfrac{\pi}{4}$

F π

3. A positive integer N has exactly six positive divisors that are perfect squares and exactly four positive divisors that are perfect cubes. In each count, 1 is included.

What is the smallest possible value of N?

A 144

B 216

C 432

D 648

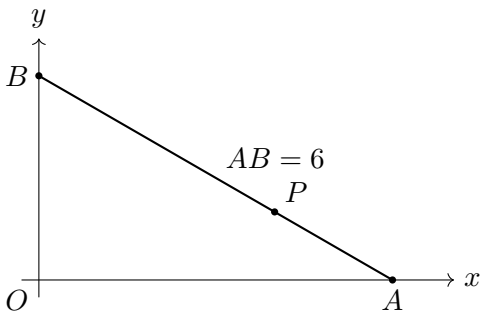
E 864

F 1024

4. The point A lies on the x-axis with non-negative x-coordinate, and the point B lies on the y-axis with non-negative y-coordinate. The length AB is 6.

The point P lies on the line segment AB, with $AP : PB = 1 : 2$. As A and B vary, P traces a curve.

What is the area, to the right of $x = 2$, bounded by this curve, the x-axis and the line $x = 2$?



One possible position of the segment is shown.

A $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$

B $\dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}$

C $\dfrac{2\pi}{3} + \sqrt{3}$

D $\dfrac{4\pi}{3} + \sqrt{3}$

E $2\pi - \sqrt{3}$

F $\dfrac{4\pi}{3} - \sqrt{3}$

5. A number is initially 0. A fair coin is tossed eight times, with independent results. After each toss, the current number is doubled if the result is heads, and increased by 1 if the result is tails.

Given that the number after the eighth toss is 15, what is the probability that the first toss was heads?

A $\frac{1}{128}$

B $\frac{1}{6}$

C $\frac{1}{3}$

D $\frac{4}{9}$

E $\frac{1}{2}$

F $\frac{2}{3}$

6. The curve C has equation

$$y = x + \frac{1}{x}, \qquad x > 0.$$

The tangent and the normal to C at a point P meet the y-axis at A and B, respectively. The origin is O, and $OA = OB$.

What is the gradient of the tangent at P?

A $\dfrac{2-\sqrt{7}}{3}$

B $\dfrac{\sqrt{7}-2}{3}$

C $\dfrac{2+\sqrt{7}}{3}$

D $\dfrac{3-\sqrt{17}}{4}$

E $\dfrac{3-\sqrt{14}}{5}$

F No such point P exists

7. For which values of the real constant k does the equation

$$\log_2(x^2 - 2x - k) = 1 + \log_2(2x - k)$$

have exactly one real solution for x?

A $k < 8$

B $0 < k < 8$

C $0 \leq k < 8$

D $0 \leq k \leq 8$

E $0 \leq k \leq 9$

F $k = 9$ only

8. Four distinct integers are chosen from

$$1, 2, 3, \ldots, 12.$$

How many different selections have a product that is a perfect square?

Selections which differ only in the order of the four integers are counted as the same selection.

A 8

B 19

C 26

D 27

E 28

F 35

9. A cubic polynomial $f(x)$ has coefficient 1 for x^3 and satisfies

$$\int_0^1 f(x)\,\mathrm{d}x = \int_1^2 f(x)\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x = 0.$$

What is the greatest possible value of

$$\int_t^{t+1} f(x)\,\mathrm{d}x$$

as t varies with $0 \leq t \leq 2$?

A 0

B $\dfrac{\sqrt{3}}{9}$

C $\dfrac{3}{8}$

D $\dfrac{25}{64}$

E $\dfrac{2\sqrt{3}}{9}$

F $\dfrac{\sqrt{3}}{3}$

10. Six distinct real numbers have mean 7, median 8 and range 9. Their smallest value is a. Which of the following is the complete set of possible values of a?

A $0 \leq a \leq 3$

B $0 < a \leq 3$

C $0 \leq a < 3$

D $0 < a < 3$

E $1 < a < 3$

F $0 < a < 4$

11. A real number r satisfies $0 < r < 1$. For each integer $n \geq 0$, the point P_n has coordinates

$$P_n = (r^n, r^{2n}).$$

A broken line joins P_0 to P_1, P_1 to P_2, and so on. The total area between this broken line and the x-axis for $0 \leq x \leq 1$ is $\frac{7}{18}$.

What is the value of

$$\sum_{n=0}^{\infty} (r^n - r^{2n})?$$

A $\frac{2}{7}$

B $\frac{2\sqrt{33}}{33}$

C $\frac{\sqrt{33}-3}{6}$

D $\frac{\sqrt{33}+3}{6}$

E $\frac{\sqrt{33}}{2}$

F $\frac{7}{2}$

12. The angles α and β satisfy

$$0 < \alpha < \frac{\pi}{2}, \qquad 0 < \beta < \frac{\pi}{2}, \qquad \tan\alpha + \tan\beta = \sqrt{2}.$$

What is the least possible value of

$$\sin^2\alpha + \sin^2\beta\,?$$

A $\dfrac{3-\sqrt{3}}{2}$

B $2-\sqrt{3}$

C $\dfrac{1}{3}$

D $\dfrac{2}{3}$

E $\sqrt{3}-1$

F There is no least value

13. In a triangle ABC, the point D lies on the side BC with $BD : DC = 3 : 2$, and the point E lies on the side CA with $CE : EA = 2 : 1$. The line segments AD and BE intersect at P.

What fraction of the area of triangle ABC is the area of triangle PDE?

A $\frac{2}{55}$

B $\frac{2}{33}$

C $\frac{4}{55}$

D $\frac{2}{15}$

E $\frac{2}{11}$

F $\frac{6}{11}$

14. The real constants a, b, c have the property that both

$$x^2 - 2ax + b \geq 0 \qquad \text{and} \qquad x^2 - 2cx + a \geq 0$$

hold for every real number x.

What is the greatest possible value of

$$a + 2c - 6b\,?$$

A $\frac{3}{8}$

B $\frac{1}{2}$

C $\frac{19}{16}$

D $\frac{5}{4}$

E There is no greatest value

F $\frac{7}{8}$

15. For $r = 0, 1, \ldots, 8$, let a_r be the coefficient of x^r in the expansion of $(1 - x)^8$.

What is the value of

$$\sum_{r=0}^{8} \frac{a_r}{(r+1)(r+2)(r+3)} \, ?$$

A $-\frac{1}{99}$

B 0

C $\frac{1}{22}$

D $\frac{1}{11}$

E $\frac{1}{9}$

F $\frac{1}{6}$

16. The positive real numbers x and y satisfy

$$x + y = \frac{9}{2}, \qquad |\log_2 x| + |\log_2 y| = 2.$$

What is the product of all the distinct values of x for which these equations have a solution?

A $\frac{81}{25}$
B 4
C 9
D $\frac{324}{25}$
E 16
F $\frac{81}{4}$

17. A particle starts at the centre P of a rectangle with side lengths 3 and 2. It moves in straight lines inside the rectangle. Whenever it reaches a side, it is reflected so that the incoming and outgoing segments make equal angles with that side. Its path never passes through a vertex.

The particle returns to P for the first time after exactly 15 reflections.

What is the least possible total distance travelled before this first return?

A $\dfrac{90}{\sqrt{13}}$

B 25

C $2\sqrt{157}$

D $\sqrt{697}$

E $4\sqrt{157}$

F Such a path is impossible

18. Six cards are numbered $1, 2, 3, 4, 5, 6$. Four cards are drawn successively without replacement, with each remaining card equally likely at each draw. Their numbers, in the order drawn, are a, b, c, d.

The curves

$$y = (x - a)(x - b) \qquad \text{and} \qquad y = (x - c)(x - d)$$

are drawn.

Given that the curves intersect, what is the probability that their point of intersection lies below the x-axis?

A $\dfrac{1}{3}$

B $\dfrac{3}{8}$

C $\dfrac{15}{38}$

D $\dfrac{5}{12}$

E $\dfrac{15}{31}$

F $\dfrac{15}{23}$

19. A cube has side length 1. Two distinct parallel planes, Π_1 and Π_2, are each perpendicular to the same body diagonal VW.

Each plane cuts the cube in a polygon of area $\dfrac{11\sqrt{3}}{16}$.

What is the perpendicular distance between the two planes?

A $\dfrac{\sqrt{3}}{12}$

B $\dfrac{1}{6}$

C $\dfrac{1}{4}$

D $\dfrac{\sqrt{3}}{6}$

E $\dfrac{\sqrt{3}}{4}$

F $\dfrac{1}{2}$

20. For each real number k, let $E(k)$ be the greatest value of

$$|x^3 - kx|$$

for $-1 \leq x \leq 1$.

What is the least possible value of $E(k)$?

A 0

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{2}{3\sqrt{3}}$

E $\frac{1}{2}$

F $\frac{3}{4}$