1. E

Write $s = a + b + c$. By the Factor Theorem,

$$P(x) = A(x-a)(x-b)(x-c), \qquad A \neq 0.$$

The three numbers $a+b$, $b+c$ and $c+a$ are distinct, and each is a root of $P(x)-4$. This is another cubic with the same leading coefficient A, so

$$P(x) - 4 = A\big(x-(a+b)\big)\big(x-(b+c)\big)\big(x-(c+a)\big).$$

The coefficient of x^2 on the left is $-As$, while on the right it is $-2As$. Therefore $s = 0$.

It follows that $a+b = -c$, $b+c = -a$ and $c+a = -b$, giving

$$P(x) - 4 = A(x+a)(x+b)(x+c) = -P(-x).$$

Thus $P(x) + P(-x) = 4$, and taking $x = 1$ gives

$$P(-1) = 4 - (-1) = \boxed{5}.$$

For completeness, the conditions are consistent: $P(x) = x^3 - 4x + 2$ has one root satisfying $-3 < x < -2$, one satisfying $0 < x < 1$, and one satisfying $1 < x < 2$, so it has three distinct real roots. Their sum is 0, and $P(-x) = 4 - P(x)$, which verifies all the stated conditions.

2. B

Let $O = (0,0)$, $P = (p,q)$ and $M = (u,v)$, and write $OP = d$. Since P lies on D, we have $2 \le d \le 4$, so both tangents exist.

The right triangles OAP and OBP are congruent, so OP passes through M and is perpendicular to AB. The right triangles OMA and OAP are similar. As $OA = 1$, this gives

$$OM \cdot OP = 1.$$

The point M lies on the ray OP, and hence

$$(u,v) = \frac{(p,q)}{p^2+q^2}, \qquad (p,q) = \frac{(u,v)}{u^2+v^2}.$$

The equation of D is $p^2 + q^2 - 6p + 8 = 0$. Substitution gives

$$1 - 6u + 8(u^2+v^2) = 0,$$

so

$$\left(u - \frac{3}{8}\right)^2 + v^2 = \frac{1}{64}.$$

Every point on this circle is nonzero and maps back by the displayed inverse formula to a point of D outside C. Thus the entire circle, rather than just a part of it, is traced. Its radius is $1/8$, and its area is $\boxed{\pi/64}$.

3. C

If the exponent of a prime in N is e, then a square divisor may use that prime with exponent $0, 2, 4, \ldots$ up to e. The choices for different primes are independent.

The factors greater than 1 in the product counting square divisors must therefore be either 6, or 3 and 2. Any prime with exponent 1 contributes no extra square or cube divisors and only increases N, so no such prime is needed for the minimum.

If one prime supplies all six square choices, its exponent is 10 or 11. Either exponent supplies four cube choices, with exponents $0, 3, 6, 9$. The minimum of this form is $2^{10} = 1024$.

Otherwise the two prime exponents are respectively 4 or 5, and 2 or 3. The first prime supplies two cube choices. To obtain four cube divisors, the second prime must also supply two cube choices, forcing its exponent to be 3. The smallest exponent pair is therefore $4, 3$.

Use the two smallest primes and give the larger exponent to the smaller prime. This yields

$$N = 2^4 \cdot 3^3 = \boxed{432} < 1024.$$

It has $3 \times 2 = 6$ square divisors and $2 \times 2 = 4$ cube divisors, so the minimum is attained.

4. F

Write $A = (a, 0)$ and $B = (0, b)$, where $a, b \geq 0$ and $a^2 + b^2 = 36$. The ratio $AP : PB = 1 : 2$ gives

$$P = (x, y) = \left(\frac{2a}{3}, \frac{b}{3}\right).$$

Consequently the complete locus is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1, \qquad x, y \geq 0.$$

Indeed, any point on this quarter-curve gives permissible endpoints $a = 3x/2$ and $b = 3y$.

Put $x = 4u$, $y = 2v$. The locus becomes the first-quadrant arc $u^2 + v^2 = 1$, while $x = 2$ becomes $u = 1/2$. The area scale factor from the (u, v)-plane to the (x, y)-plane is $4 \times 2 = 8$.

The required unit-circle region is a sector of angle $\pi/3$ with a right triangle removed. Its area is

$$\frac{1}{2} \cdot \frac{\pi}{3} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}.$$

The requested area is therefore

$$8\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) = \boxed{\frac{4\pi}{3} - \sqrt{3}}.$$

5. C

Every sequence of eight tosses has probability 2^{-8}. We count the sequences ending at 15.

Suppose there are h heads. Each tail adds 1, which is subsequently doubled once for each later head. Its contribution to the final number is therefore one of $1, 2, 4, \ldots, 2^h$. There are $8 - h$ such contributions. Conversely, once the contributions are listed in decreasing order, they determine the toss sequence uniquely: the exponent attached to a tail is the number of heads still to come.

There must be at least four tails. Indeed, 15 cannot be the sum of at most three powers of 2: all terms are at most 8; if there is one term equal to 1, the other two would have to sum to 14, which is impossible; with three terms equal to 1 the sum is only 3. This also rules out fewer than three terms. Hence $h \leq 4$. If $h = 0$, the final number is 8; if $h = 1$, seven contributions of at most 2 cannot total 15. Thus h is 2, 3 or 4.

The complete possibilities are

h	contributions of the tails	first toss heads?
2	$4+4+4+1+1+1$	no
2	$4+4+2+2+2+1$	no
3	$8+4+1+1+1$	no
3	$8+2+2+2+1$	no
3	$4+4+4+2+1$	yes
4	$8+4+2+1$	yes

For an explicit check of completeness, when $h = 2$ let the numbers of 1's, 2's and 4's be r, s, t. Then $r+s+t = 6$ and $r+2s+4t = 15$, so $s+3t = 9$; only $(r, s, t) = (3, 0, 3)$ and $(1, 3, 2)$ are nonnegative solutions. For $h = 3$, five contributions totalling 15 can contain either one 8 or none. With an 8, the four remaining terms total 7, giving $4+1+1+1$ or $2+2+2+1$. Without an 8, the only choice is $4+4+4+2+1$. For $h = 4$, four powers of 2 summing to 15 must be $8, 4, 2, 1$.

The first toss is heads exactly when the greatest contribution is smaller than 2^h, since then at least one head occurs before the first tail. This happens in two of the six equally likely sequences. The required conditional probability is $\boxed{2/6 = 1/3}$.

6. A

Write $P = (t, t + 1/t)$ with $t > 0$, and let the tangent gradient be m. The power rule gives

$$m = 1 - \frac{1}{t^2} < 1.$$

The tangent's y-intercept is

$$t + \frac{1}{t} - mt = \frac{2}{t} > 0.$$

If $m = 0$, the normal is vertical and does not meet the y-axis, so this case is excluded. Also, the tangent and normal cannot have the same y-intercept, since these two distinct lines already meet at P, away from the y-axis. Thus $OA = OB$ forces the normal's intercept to be $-2/t$.

The normal gradient is consequently

$$\frac{t + 1/t + 2/t}{t} = 1 + \frac{3}{t^2} = 4 - 3m.$$

The tangent and normal are perpendicular, giving

$$m(4 - 3m) = -1, \qquad 3m^2 - 4m - 1 = 0.$$

Hence $m = (2 \pm \sqrt{7})/3$. The positive choice exceeds 1 and is impossible. The other choice gives $t^2 = (\sqrt{7} - 1)/2 > 0$ and a nonzero m, so it does yield a valid point with the stated opposite intercepts. Therefore

$$\boxed{m = \frac{2 - \sqrt{7}}{3}}.$$

7. C

Both logarithmic arguments must be positive. Where the logarithms are defined, their equality is equivalent to

$$x^2 - 2x - k = 2(2x - k), \qquad x^2 - 6x + k = 0.$$

At an algebraic root we have $k = 6x - x^2$, so

$$2x - k = x^2 - 4x = x(x - 4).$$

The first logarithmic argument is twice the second. Thus an algebraic root is a genuine solution exactly when $x < 0$ or $x > 4$.

For $k \leq 9$, the algebraic roots are $3 \pm \sqrt{9 - k}$; for $k > 9$, no real algebraic root exists.

- If $k < 0$, the smaller root is negative and the larger root exceeds 6, so there are two genuine solutions.
- If $0 \leq k < 8$, the smaller root is between 0 and 3 and is invalid, while the larger root exceeds 4 and is valid. There is exactly one solution.
- If $8 \leq k \leq 9$, both roots lie between 2 and 4, so neither is valid. At $k = 8$, the root 4 gives logarithms of zero; at $k = 9$, the repeated root 3 is also invalid.
- If $k > 9$, there are no real roots.

At $k = 0$, the root 0 is invalid but 6 is valid, so the lower endpoint is included. The complete answer is $\boxed{0 \leq k < 8}$.

8. D

A positive integer is a perfect square exactly when every prime in its factorisation has an even exponent.

Neither 7 nor 11 can be chosen: each is the only available multiple of its prime. Also, either both 5 and 10 are chosen, or neither is chosen, because these are the only available multiples of 5.

The remaining eight integers may be grouped according to what remains after removing square factors:

Form, with s a positive integer	Available integers
s^2	$1, 4, 9$
$2s^2$	$2, 8$
$3s^2$	$3, 12$
$6s^2$	6

Case 1: neither 5 **nor** 10 **is chosen.**

If 6 is not chosen, an even number must be selected from each of the groups $2, 8$ and $3, 12$. Four squares cannot be selected because only three are available. The possibilities are therefore:

- both $2, 8$ and two of $1, 4, 9$: 3 selections;
- both $3, 12$ and two of $1, 4, 9$: 3 selections;
- all four of $2, 8, 3, 12$: 1 selection.

If 6 is chosen, the exponents of both 2 and 3 must be made even. Exactly one of $2, 8$ and exactly one of $3, 12$ must then be chosen, together with one of $1, 4, 9$. This gives $2 \cdot 2 \cdot 3 = 12$ selections.

Case 1 therefore contributes $3 + 3 + 1 + 12 = 19$ selections.

Case 2: both 5 **and** 10 **are chosen.**

Their product is $50 = 2 \cdot 5^2$, so the product of the other two integers must be twice a square. From the table, the only possibilities are:

- one of $1, 4, 9$ and one of $2, 8$: $3 \cdot 2 = 6$ selections;
- 6 and one of $3, 12$: 2 selections.

Thus Case 2 contributes 8 selections. The cases are disjoint and exhaustive, giving a total of

$$19 + 8 = \boxed{27}.$$

No ordering factors are needed, since each count specifies an unordered selection of four distinct integers.

9. E

Let F be a polynomial antiderivative of f, and put

$$I(t) = F(t+1) - F(t) = \int_t^{t+1} f(x)\,\mathrm{d}x.$$

The leading term of $F(x)$ is $x^4/4$. In the difference $F(t+1) - F(t)$ the fourth-degree terms cancel, and the leading term is t^3. Thus I is a cubic polynomial with leading coefficient 1.

The three given integrals say that $I(0) = I(1) = I(2) = 0$. The Factor Theorem therefore gives

$$I(t) = t(t-1)(t-2) = (t-1)^3 - (t-1).$$

Consequently

$$I'(t) = 3(t-1)^2 - 1 = 0 \quad \Longleftrightarrow \quad t = 1 \pm \frac{1}{\sqrt{3}}.$$

Both stationary points lie between 0 and 2. The function is positive for $0 < t < 1$ and negative for $1 < t < 2$, while $I(0) = I(2) = 0$. Hence the greatest value occurs at $t = 1 - 1/\sqrt{3}$, and it is

$$-\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

The conditions on f are consistent: one may take

$$F(x) = \frac{1}{4}x(x-1)(x-2)(x-3), \qquad f(x) = x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2}.$$

This also directly verifies the three prescribed integrals.

10. D

Write the six numbers in increasing order as

$$a < b < c < d < e < f.$$

The conditions give $a + b + c + d + e + f = 42$, $c + d = 16$ and $f = a + 9$. Thus

$$b + e = 17 - 2a, \qquad c < 8 < d.$$

Since $b > a$ and $e > d > 8$, we have $17 - 2a > a + 8$, so $a < 3$. Since $b < c < 8$ and $e < f = a + 9$, we have $17 - 2a < a + 17$, so $a > 0$. Both inequalities are strict because all six observations are distinct.

To check that no further restriction is needed, take any a with $0 < a < 3$ and set $\varepsilon = a(3 - a)/12$. Consider

$$a, \quad 8 - \frac{5a}{3}, \quad 8 - \varepsilon, \quad 8 + \varepsilon, \quad 9 - \frac{a}{3}, \quad a + 9.$$

Their sum is 42, their middle two have sum 16, and their largest minus their smallest is 9. Their consecutive differences are

$$\frac{8(3 - a)}{3}, \quad \frac{a(a + 17)}{12}, \quad \frac{a(3 - a)}{6}, \quad \frac{(4 - a)(3 - a)}{12}, \quad \frac{4a}{3},$$

all of which are positive for $0 < a < 3$. Hence these really are six distinct numbers in the displayed order, and every such a is attainable. The complete answer is $0 < a < 3$.

11. B

The region beneath the segment P_nP_{n+1} is a trapezium. Its width is $r^n - r^{n+1}$, and its two vertical sides have lengths r^{2n} and r^{2n+2}. Its area is therefore

$$\frac{1}{2}(r^n - r^{n+1})(r^{2n} + r^{2n+2}) = \frac{1}{2}(1-r)(1+r^2)r^{3n}.$$

These areas form a convergent geometric series with common ratio r^3. Their sum gives

$$\frac{(1-r)(1+r^2)}{2(1-r^3)} = \frac{1+r^2}{2(1+r+r^2)} = \frac{7}{18}.$$

Cancelling $1-r$ is valid because $0 < r < 1$. Rearrangement yields

$$2r^2 - 7r + 2 = 0, \qquad r + \frac{1}{r} = \frac{7}{2}.$$

The requested series is the difference of two convergent geometric series:

$$\frac{1}{1-r} - \frac{1}{1-r^2} = \frac{r}{1-r^2} = \frac{1}{1/r - r}.$$

Since $0 < r < 1$, the denominator is positive. Also

$$\left(\frac{1}{r} - r\right)^2 = \left(\frac{1}{r} + r\right)^2 - 4 = \frac{33}{4}.$$

Thus the required sum is $2/\sqrt{33} = 2\sqrt{33}/33$.

There is exactly one admissible value of r, namely $(7 - \sqrt{33})/4$; the other quadratic root is greater than 1. The admissible root is strictly between 0 and 1, so the geometric sums and the area construction are valid.

12. A

Put $p = \tan\alpha$ and $q = \tan\beta$. Then $p, q > 0$ and $p + q = \sqrt{2}$. Write $r = pq$. Since

$$(p - q)^2 = 2 - 4r \geq 0,$$

we have $0 < r \leq 1/2$.

From $\tan\theta = \sin\theta/\cos\theta$ and $\sin^2\theta + \cos^2\theta = 1$,

$$\sin^2\alpha = \frac{p^2}{1+p^2}, \qquad \sin^2\beta = \frac{q^2}{1+q^2}.$$

Hence, on writing $m = \sin^2\alpha + \sin^2\beta$ and using $p^2 + q^2 = 2 - 2r$,

$$m = \frac{p^2 + q^2 + 2p^2q^2}{1 + p^2 + q^2 + p^2q^2} = \frac{2r^2 - 2r + 2}{r^2 - 2r + 3}.$$

The denominator is $(r-1)^2 + 2 > 0$. Rearranging gives

$$(2 - m)r^2 + (2m - 2)r + (2 - 3m) = 0.$$

As $0 < m < 2$, this is a quadratic in r. Its discriminant must be nonnegative:

$$(2m - 2)^2 - 4(2 - m)(2 - 3m) = 4(-2m^2 + 6m - 3) \geq 0.$$

Therefore

$$m \geq \frac{3 - \sqrt{3}}{2}.$$

It remains to check that this lower bound is attainable with acute angles. At equality, the repeated root in r is $r = 2 - \sqrt{3}$, which satisfies $0 < r < 1/2$. The quadratic

$$z^2 - \sqrt{2}z + (2 - \sqrt{3}) = 0$$

has two positive real roots: its discriminant is $4\sqrt{3} - 6 > 0$, its sum of roots is positive, and its product is positive. Taking these roots as p, q gives positive tangents of two acute angles and attains the bound. Thus the least value really is $(3 - \sqrt{3})/2$.

In particular, setting $\alpha = \beta$ would give $2/3$, which is not the minimum.

13. C

Write $[XYZ]$ for the area of triangle XYZ, and put

$$U = [ABP], \qquad V = [BCP], \qquad W = [CAP].$$

First, triangles BPD and CPD have the same perpendicular height to the line BC, so their areas are in the ratio $BD : DC = 3 : 2$. Also, $[ABP]/[BPD] = AP/PD$ and $[ACP]/[CPD] = AP/PD$, because each of these pairs has bases on the line AD and a common height. It follows that $U : W = 3 : 2$.

Similarly, triangles APE and CPE have bases on AC and a common height, so their areas are in the ratio $AE : EC = 1 : 2$. Each is obtained from the corresponding triangle APB or CPB by multiplying its area by PE/PB. Consequently $U : V = 1 : 2$.

Combining the two ratios gives

$$U : V : W = 3 : 6 : 2.$$

Since these three triangles partition ABC, we get $[ABP]/[ABC] = 3/11$. Also $[ABD]/[ABC] = BD/BC = 3/5$. Using bases AP and AD on the same line and the common vertex B,

$$\frac{AP}{AD} = \frac{[ABP]}{[ABD]} = \frac{3/11}{3/5} = \frac{5}{11}, \quad \text{so} \quad \frac{PD}{AD} = \frac{6}{11}.$$

Finally,

$$\frac{[ADE]}{[ABC]} = \frac{AE}{AC}\,\frac{[ADC]}{[ABC]} = \frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$$

Triangles PDE and ADE have bases PD and AD on the same line and the common vertex E. Thus

$$\frac{[PDE]}{[ABC]} = \frac{6}{11} \cdot \frac{2}{15} = \frac{4}{55}.$$

14. F

Completing the square in the two given expressions shows that the conditions are equivalent to

$$b \geq a^2, \qquad a \geq c^2.$$

In particular $a \geq 0$ and $c \leq \sqrt{a}$. Therefore

$$a + 2c - 6b \leq a + 2\sqrt{a} - 6a^2.$$

Set $u = \sqrt{a}$, so that $u \geq 0$. The expression on the right becomes

$$h(u) = u^2 + 2u - 6u^4.$$

Differentiate and factor:

$$h'(u) = 2u + 2 - 24u^3 = -2(2u - 1)(6u^2 + 3u + 1).$$

For $u \geq 0$ the second factor is positive. Thus h increases up to $u = 1/2$ and decreases thereafter, and its greatest value is

$$h\left(\frac{1}{2}\right) = \frac{1}{4} + 1 - \frac{6}{16} = \frac{7}{8}.$$

This bound is attained when

$$a = \frac{1}{4}, \qquad b = \frac{1}{16}, \qquad c = \frac{1}{2}.$$

Indeed, the two given quadratics then become $(x - 1/4)^2$ and $(x - 1/2)^2$, so both conditions hold for every real x. The greatest value is consequently $7/8$.

An entirely algebraic check of the sharp bound is

$$\frac{7}{8} - h(u) = \frac{(2u - 1)^2(12u^2 + 12u + 7)}{8} \geq 0 \qquad (u \geq 0).$$

15. C

By the binomial theorem, $a_r = (-1)^r \binom{8}{r}$. The factorial definitions give

$$\frac{\binom{8}{r}}{(r+1)(r+2)(r+3)} = \frac{1}{9 \cdot 10 \cdot 11} \binom{11}{r+3}.$$

Consequently, if the requested sum is S, then

$$990S = \binom{11}{3} - \binom{11}{4} + \cdots + \binom{11}{11}.$$

The complete alternating binomial sum is $(1-1)^{11} = 0$, so

$$990S = 1 - 11 + \binom{11}{2} = 1 - 11 + 55 = 45.$$

Thus $S = 45/990 = 1/22$.

An independent way to see the cancellation is to note that, for each non-negative integer r,

$$\int_0^1 t^r (1-t)^2 \, \mathrm{d}t = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3} = \frac{2}{(r+1)(r+2)(r+3)}.$$

Combining the finitely many integrals therefore gives

$$S = \frac{1}{2} \int_0^1 (1-t)^2 \sum_{r=0}^{8} a_r t^r \, \mathrm{d}t = \frac{1}{2} \int_0^1 (1-t)^{10} \, \mathrm{d}t = \frac{1}{22}.$$

16. D

Both x and y cannot be at most 1, since their sum is $9/2$.

If $x \geq 1$ and $y \geq 1$, the logarithmic condition becomes $\log_2(xy) = 2$, so $xy = 4$. The possible values of x are the two roots of

$$x^2 - \frac{9}{2}x + 4 = 0,$$

namely $(9 - \sqrt{17})/4$ and $(9 + \sqrt{17})/4$. They are distinct and both exceed 1, since $\sqrt{17} < 5$. In either solution y is the other root, so both solutions are admissible. Their x-values have product 4.

If $x \geq 1$ and $0 < y \leq 1$, the condition becomes $\log_2(x/y) = 2$. Thus $x = 4y$, giving

$$(x, y) = \left(\frac{18}{5}, \frac{9}{10}\right).$$

These values satisfy the case restrictions. Interchanging the roles of x and y gives the remaining solution

$$(x, y) = \left(\frac{9}{10}, \frac{18}{5}\right).$$

There is no boundary solution with $x = 1$ or $y = 1$, since the other variable would have to be both 4 and $7/2$. The four solutions therefore belong to disjoint cases, and their four x-values are distinct (the first equation determines y uniquely from x).

The requested product is

$$4 \cdot \frac{18}{5} \cdot \frac{9}{10} = \frac{324}{25}.$$

17. C

At a reflection, reflect the rectangle instead of the moving segment. The equal-angle rule means that the unfolded path continues in a straight line through a tiling by congruent rectangles.

Take the side of length 3 as horizontal. The centres of the reflected rectangles form a rectangular grid with horizontal spacing 3 and vertical spacing 2. A return to P corresponds to reaching one of these centres. By symmetry, its displacement from the starting centre may be written as $(3m, 2n)$, where m, n are non-negative integers, not both zero.

The straight segment crosses m vertical sides and n horizontal sides. Vertex contacts are excluded, so the number of reflections is $m + n$. Thus

$$m + n = 15, \qquad L^2 = 9m^2 + 4n^2.$$

The return is the first return precisely when m and n have no common factor greater than 1. Indeed, a common factor would give an earlier grid centre on the same segment; conversely, an earlier centre would have integer displacement $(3a, 2b)$ in the same direction and therefore would give a common factor. Hence m and n must be coprime.

Substituting $n = 15 - m$ gives

$$L^2 = 13m^2 - 120m + 900 = 13\left(m - \frac{60}{13}\right)^2 + \frac{8100}{13}.$$

The closest integer to $60/13$ is 5, giving $(m, n) = (5, 10)$, but that pair is not coprime and returns to P earlier. The next closest integer is 4, giving $(m, n) = (4, 11)$ and $L^2 = 628$. Every integer $m \leq 3$ or $m \geq 6$ is farther from $60/13$, so none gives a smaller value.

The pair $(4, 11)$ is coprime. Its path also avoids vertices: if the unfolded segment met a grid vertex, then for some real t, both $4t$ and $11t$ would be an integer plus $1/2$. Thus $8t$ and $22t$ would both be odd integers. This is impossible, since $11(8t) = 4(22t)$ would equate an odd integer with an even one. The proposed path is therefore admissible and has no earlier return.

It follows that the least distance is $\sqrt{628} = 2\sqrt{157}$. No theorem about billiards or lattice paths is assumed: the reflection, common-factor and Pythagoras arguments above establish everything required.

18. C

Every ordered draw has probability $1/(6 \cdot 5 \cdot 4 \cdot 3)$. Group the draws by their selected four cards and by their division into two unordered pairs. There are 15 selections of four cards and 3 pairings for each selection, giving 45 equally likely pairings. Each represents 8 ordered draws, accounting for order within the two pairs and order of the pairs.

Write the selected numbers in increasing order as $r < s < t < u$. Of the three pairings:

- (r, s), (t, u) have disjoint negative regions, so the curves cannot meet below the axis.
- (r, u), (s, t) have nested negative regions. Their difference is linear and is strictly negative at both s and t, so it cannot be zero between those two values. They cannot meet in their common negative region.
- (r, t), (s, u) interlace. Their linear difference is negative at s and positive at t, so the unique intersection lies between s and t, where both curves are negative.

Exactly one pairing for each selection is favourable, giving 15 favourable pairings.

Subtracting the two quadratic equations shows that the curves fail to intersect exactly when $a+b = c+d$. In that case $ab \neq cd$, since equal sums and products would make the two root pairs identical, contrary to all four cards being distinct.

The pair sums allowing two different pairs are:

sum	pairs	choices of two pairs
5	$(1,4),(2,3)$	1
6	$(1,5),(2,4)$	1
7	$(1,6),(2,5),(3,4)$	3
8	$(2,6),(3,5)$	1
9	$(3,6),(4,5)$	1

Pairs of the same sum are automatically disjoint. There are therefore 7 non-intersecting pairings, leaving 38 intersecting pairings. The conditional probability is $15/38$.

19. D

Take V as one vertex and extend the three edges issuing from V. A plane perpendicular to the body diagonal cuts these lines at equal distances a from V: rotation by 120° about that diagonal fixes the plane and permutes the three edge lines.

The intercept triangle is equilateral, with side $a\sqrt{2}$ and area $\sqrt{3}a^2/2$. For $0 \le a \le 1$, the section is this entire triangle, so its area is at most $\sqrt{3}/2$. Reflection in the cube centre gives the same upper bound for $2 \le a \le 3$. The required area $11\sqrt{3}/16$ is larger than $\sqrt{3}/2$, so both sections have $1 < a < 2$.

In this case the three opposite faces remove three non-overlapping equilateral corner triangles from the intercept triangle. Each has side $(a-1)\sqrt{2}$, by similarity or by measuring along the two edge directions. Hence the section area is

$$A(a) = \frac{\sqrt{3}}{2}\left[a^2 - 3(a-1)^2\right].$$

The given area implies

$$a^2 - 3(a-1)^2 = \frac{11}{8}, \qquad 16a^2 - 48a + 35 = 0,$$

so $a = 5/4$ or $a = 7/4$. Both lie in the required range and they are the only possible sections.

The perpendicular distance from V to the plane is $a/\sqrt{3}$. For an elementary derivation, the foot is the centre of the equilateral intercept triangle; its distance to an intercept vertex is $a\sqrt{2/3}$. Pythagoras, with hypotenuse a, gives the perpendicular distance $\sqrt{a^2 - 2a^2/3} = a/\sqrt{3}$.

The two-plane separation is therefore

$$\frac{7/4 - 5/4}{\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

20. B

Put $p_k(x) = x^3 - kx$ and $E = E(k)$. For every real k,

$$p_k(1) - 2p_k\left(\frac{1}{2}\right) = (1-k) - 2\left(\frac{1}{8} - \frac{k}{2}\right) = \frac{3}{4}.$$

Both $p_k(1) \le E$ and $-p_k(1/2) \le E$, so $3/4 \le 3E$. Thus $E(k) \ge 1/4$ for every real k.

Choose $k = 3/4$. Then

$$p_k'(x) = 3x^2 - \frac{3}{4},$$

so the only stationary inputs are $-1/2$ and $1/2$. At these inputs and the endpoints,

$$p_k(-1) = -\frac{1}{4}, \quad p_k(-1/2) = \frac{1}{4}, \quad p_k(1/2) = -\frac{1}{4}, \quad p_k(1) = \frac{1}{4}.$$

The derivative has constant sign between consecutive listed inputs. These values therefore give the global maximum and minimum on $-1 \le x \le 1$, and $E(3/4) = 1/4$. The least possible value is exactly $1/4$.

The minimising coefficient is unique: if $k < 3/4$, then $p_k(1) > 1/4$; if $k > 3/4$, then $p_k(1/2) < -1/4$.

To discover the candidate rather than guess it, write an interior stationary input as $t > 0$, giving $k = 3t^2$ and interior error $2t^3$. Balancing this with the endpoint error leads to

$$1 - 3t^2 = 2t^3, \qquad (t+1)^2(2t-1) = 0,$$

so $t = 1/2$ and $k = 3/4$. The preceding two-input bound, not the balance alone, proves global optimality.